Adaptivity and Universality: Problem-dependent Universal Regret for Online Convex Optimization

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Abstract

Universal online learning aims to achieve optimal regret guarantees without requiring prior knowledge of the curvature of online functions. Existing methods have established minimax-optimal regret bounds for universal online learning, where a *single* algorithm can simultaneously attain $\mathcal{O}(\sqrt{T})$ regret for convex functions, $\mathcal{O}(d \log T)$ for exp-concave functions, and $\mathcal{O}(\log T)$ for strongly convex functions, where T is the number of rounds and d is the dimension of the feasible domain. However, these methods still lack problem-dependent adaptivity. In particular, no universal method provides regret bounds that scale with the *gradient variation* V_T, a key quantity that plays a crucial role in applications such as stochastic optimization and fast-rate convergence in games. In this work, we introduce UniGrad, a novel approach that achieves both universality and adaptivity, with two distinct realizations: UniGrad.Correct and UniGrad.Bregman. Both methods achieve universal regret guarantees that adapt to gradient variation, simultaneously attaining $\mathcal{O}(\log V_T)$ regret for strongly convex functions and $\mathcal{O}(d \log V_T)$ regret for exp-concave functions. For convex functions, the regret bounds differ: UniGrad.Correct achieves an $\mathcal{O}(\sqrt{V_T \log V_T})$ bound while preserving the RVU property that is crucial for fast convergence in online games, whereas UniGrad.Bregman achieves the optimal $\mathcal{O}(\sqrt{V_T})$ regret bound through a novel design. Both methods employ a meta algorithm with $\mathcal{O}(\log T)$ base learners, which naturally requires $\mathcal{O}(\log T)$ gradient queries per round. To further enhance computational efficiency, we introduce UniGrad++, which retains the regret guarantees while reducing the gradient query requirement to just 1 per round via a surrogate optimization technique. Our results advance the state-of-the-art in universal online learning, with immediate implications and applications, including small-loss and gradient-variance bounds, novel guarantees for the stochastically extended adversarial model, and faster convergence in online games. Additionally, as an extension, we provide an anytime variant of our method.

1. Introduction

Online convex optimization (OCO) is a versatile and powerful framework for modeling the interaction between a learner and the environment over time (Hazan, 2016; Orabona, 2019). In each round $t \in [T]$, the learner selects a decision $\mathbf{x}_t$ from a convex compact set $\mathcal{X} \subseteq \mathbb{R}^d$, while the environment simultaneously chooses a convex online function $f_t : \mathcal{X} \to \mathbb{R}$. The learner then incurs a loss $f_t(\mathbf{x}_t)$ and receives information about the online function to update the decision to $\mathbf{x}_{t+1}$, with the goal of optimizing the game-theoretic performance metric known as *regret* (Cesa-Bianchi and Lugosi, 2006), whose definition is given by

$$\textsc{Reg}_T = \sum_{t=1}^{T} f_t(\mathbf{x}_t) - \min_{\mathbf{x}\in\mathcal{X}} \sum_{t=1}^{T} f_t(\mathbf{x}). \tag{1.1}$$

The regret quantifies the learner's excess loss relative to the best offline decision in hindsight. In OCO, it is well-established that the type and curvature of online functions significantly influence the minimax regret bounds. For convex functions, Online Gradient Descent (OGD) can achieve an $\mathcal{O}(\sqrt{T})$ regret (Zinkevich, 2003). For α-exp-concave functions, Online Newton Step (ONS), with prior knowledge of the curvature coefficient α, attains an $\mathcal{O}(\frac{d}{\alpha}\log T)$ regret (Hazan et al., 2007). For λ-strongly convex functions, OGD with a suitable step size configuration relating to the curvature coefficient λ enjoys an $\mathcal{O}(\frac{1}{\lambda}\log T)$ regret (Hazan et al., 2007). These regret rates have been proved to be minimax optimal (Ordentlich and Cover, 1998; Abernethy et al., 2008).

Notably, there are two caveats in the above results. First, to achieve the optimal regret bounds, these algorithms require prior knowledge of the function type and the parameter characterizing the curvature as the algorithmic input. Moreover, these algorithms are only optimal in the worst case and lack adaptivity to problem-dependent hardness. Therefore, modern online learning research strengthens these results in two key aspects: *(i) universality*, to handle the unknown types and curvatures; *(ii) adaptivity*, to adapt to the problem-dependent hardness. In the following, we discuss each aspect in detail.

1.1 Universality: Unknown Curvature of Online Functions

Traditionally, to attain the minimax optimality, the learner must select the "correct" algorithm with well-tuned parameters tailored to each specific class of online functions, which requires the prior knowledge of the curvature information and can be burdensome in practice. *Universal online learning* aims to develop a single algorithm agnostic to the specific function type and curvature while achieving the same regret guarantees as if they were known (van Erven and Koolen, 2016; Cutkosky and Boahen, 2017; Wang et al., 2019; Mhammedi et al., 2019; Zhang et al., 2021, 2022a; Yan et al., 2023, 2024; Yang et al., 2024). The pioneering work of van Erven and Koolen (2016) proposes the MetaGrad algorithm that achieves an $\mathcal{O}(\sqrt{T})$ regret for convex functions and an $\mathcal{O}(\frac{d}{\alpha}\log T)$ regret for exp-concave/strongly convex functions, leaving a gap to the optimal $\mathcal{O}(\frac{1}{\lambda}\log T)$ regret for strongly convex functions, which is later closed by Wang et al. (2019).

The above methods rely on a meta-base two-layer structure to handle the uncertainty of the function curvature. Specifically, for each function family, the online learner maintains a set of base learners with different configurations, with a meta algorithm running on top to combine their outputs. To achieve the desired universality, MetaGrad and its variants require the base learners to optimize over heterogeneous surrogate loss functions customized to each function family, and use a complex meta algorithm to adaptively combine these heterogeneous base learners. The entire structure can be complex and technically challenging to analyze due to the heterogeneity of the base learners.

To enhance flexibility, Zhang et al. (2022a) introduce a simple and general framework that uses a meta algorithm equipped with a second-order regret bound, directly operating the base learners on the original online functions. The method still achieves optimal regret bounds universally: $\mathcal{O}(\sqrt{T})$ for convex functions, $\mathcal{O}(\frac{d}{\alpha}\log T)$ for α-exp-concave functions, and $\mathcal{O}(\frac{1}{\lambda}\log T)$ for λ-strongly convex functions. It requires $\mathcal{O}(\log T)$ gradient queries per round, as $\mathcal{O}(\log T)$ base learners are maintained in parallel with different configurations.

1.2 Adaptivity: Unknown Niceness of Online Environments

Within a specific function family, the algorithm's performance is also influenced by the problem-dependent hardness. Ideally, a well-designed online algorithm should be robust to the worst-case scenarios while simultaneously delivering faster-rate guarantees in more benign and easier environments. *Problem-dependent online learning* aims to design algorithms with regret guarantees scaling with the problem-dependent quantities (Srebro et al., 2010; Chiang et al., 2012; Orabona et al., 2012; Wei and Luo, 2018; Zhang et al., 2019; Cutkosky, 2020; Zhao et al., 2024). There are several different notions of problem-dependent adaptivity, including the small-loss bound, the gradient-variance bound, and the gradient-variation bound. Among these, we focus on *gradient-variation regret* (Chiang et al., 2012; Yang et al., 2014), due to its fundamental importance in modern online learning and its strong connections to game theory and optimization. The gradient-variation regret replaces the dependence on the time horizon T with the gradient-variation quantity V_T defined as

$$V_T \triangleq \sum_{t=2}^{T} \sup_{\mathbf{x}\in\mathcal{X}} \|\nabla f_t(\mathbf{x}) - \nabla f_{t-1}(\mathbf{x})\|^2, \tag{1.2}$$

which measures the cumulative change of gradients across consecutive functions. When the online functions change gradually, V_T can be very small, even as low as $\mathcal{O}(1)$ when the functions are basically the same. On the other hand, under the standard bounded-gradient assumption, V_T is at most $\mathcal{O}(T)$ in the worst case. Under smoothness, more adaptive algorithms can improve the minimax regret to $\mathcal{O}(\sqrt{V_T})$, $\mathcal{O}(\frac{d}{\alpha}\log V_T)$, and $\mathcal{O}(\frac{1}{\lambda}\log V_T)$ for convex, α-exp-concave, and λ-strongly convex functions, respectively.

These results are important since they safeguard the minimax worst-case rates and can be much better when the environment is easier such as $V_T = \mathcal{O}(1)$. Moreover, as demonstrated by Zhao et al. (2024), the gradient-variation regret is more fundamental than other well-known problem-dependent quantities like the small loss $F_T = \min_{\mathbf{x}\in\mathcal{X}} \sum_{t=1}^{T} f_t(\mathbf{x})$ (Srebro et al., 2010; Orabona et al., 2012), since gradient-variation regret can imply small-loss bounds directly in analysis. Furthermore, gradient variation plays a crucial role in bridging adversarial and stochastic optimization (Sachs et al., 2022; Chen et al., 2024), enabling fast rates in online games (Syrgkanis et al., 2015; Zhang et al., 2022b), and facilitating acceleration in smooth offline optimization (Zhao, 2025; Zhao et al., 2025). As a result, there has been a surge of recent interest in achieving gradient-variation regret bounds in various online learning problems (Zhao et al., 2020; Sachs et al., 2022; Zhang et al., 2022b; Qiu et al., 2023; Tsai et al., 2023; Zhao et al., 2024; Tarzanagh et al., 2024; Xie et al., 2024).

1.3 Our Contributions and Techniques

Motivated by the above progress of modern online learning, a natural question arises: *Is it possible to design a single algorithm that achieves both universality and adaptivity?* More concretely, the goal is to design a universal algorithm with gradient-variation regret bounds across different function families: a single online algorithm simultaneously achieving an

Table 1: Comparison with existing results (including prior works and our conference version (Yan et al., 2023)). The second column shows the regret bounds for strongly convex, exp-concave, and convex functions. The third column indicates the computational efficiency, where "# Grad." is the number of gradient queries in each round and "# Base" is the number of maintained base learners. The last column indicates whether the method can support the RVU property that is vital for the fast-convergence of online games.

Method	**Regret Bounds**			**Efficiency**		**RVU**
	Strongly Convex	Exp-concave	Convex	# Grad.	# Base	
van Erven and Koolen (2016)	$\mathcal{O}(d \log T)$	$\mathcal{O}(d \log T)$	$\mathcal{O}(\sqrt{T})$	1	$\mathcal{O}(\log T)$	✗
Wang et al. (2019)	$\mathcal{O}(\log T)$	$\mathcal{O}(d \log T)$	$\mathcal{O}(\sqrt{T})$	1	$\mathcal{O}(\log T)$	✗
Zhang et al. (2022a)	$\mathcal{O}(\log V_T)$	$\mathcal{O}(d \log V_T)$	$\mathcal{O}(\sqrt{T})$	$\mathcal{O}(\log T)$	$\mathcal{O}(\log T)$	✗
Yan et al. (2023)	$\mathcal{O}(\log V_T)$	$\mathcal{O}(d \log V_T)$	$\mathcal{O}(\sqrt{V_T \log V_T})$	$\mathcal{O}((\log T)^2)$	$\mathcal{O}((\log T)^2)$	✓
UniGrad.Correct (Theorem 1)	$\mathcal{O}(\log V_T)$	$\mathcal{O}(d \log V_T)$	$\mathcal{O}(\sqrt{V_T \log V_T})$	$\mathcal{O}(\log T)$	$\mathcal{O}(\log T)$	✓
UniGrad.Bregman (Theorem 2)	$\mathcal{O}(\log V_T)$	$\mathcal{O}(d \log V_T)$	$\mathcal{O}(\sqrt{V_T})$	$\mathcal{O}(\log T)$	$\mathcal{O}(\log T)$	✗
UniGrad++.Correct (Theorem 3)	$\mathcal{O}(\log V_T)$	$\mathcal{O}(d \log V_T)$	$\mathcal{O}(\sqrt{V_T \log V_T})$	1	$\mathcal{O}(\log T)$	✓
UniGrad++.Bregman (Theorem 4)	$\mathcal{O}(\log V_T)$	$\mathcal{O}(d \log V_T)$	$\mathcal{O}(\sqrt{V_T})$	1	$\mathcal{O}(\log T)$	✗

$\mathcal{O}(\sqrt{V_T})$ regret for convex functions, an $\mathcal{O}(\frac{d}{\alpha} \log V_T)$ regret for α-exp-concave functions, and an $\mathcal{O}(\frac{1}{\lambda} \log V_T)$ regret for λ-strongly convex functions, respectively.

Zhang et al. (2022a) obtain partial results with regret bounds of $\mathcal{O}(\sqrt{T})$, $\mathcal{O}(\frac{d}{\alpha} \log V_T)$, and $\mathcal{O}(\frac{1}{\lambda} \log V_T)$ for convex, α-exp-concave, and λ-strongly convex functions, simultaneously. Their result, however, falls short in the convex case, which is arguably the most important: the improvement from T to V_T is polynomial for convex functions, whereas logarithmic for the other cases. This gap was left open in their work.

In this paper, we resolve the open problem and obtain the desired gradient-variation regret bounds for all three types of functions. Our approach builds on the *optimistic online ensemble* framework developed for gradient-variation dynamic regret (Zhao et al., 2024). However, significant new ingredients are required for universal online learning, particularly in properly encoding gradient-variation adaptivity across all three types of functions. We propose a novel approach called UniGrad (short for "Universal Gradient-variation Online Learning"), consisting of two distinct realizations based on fundamentally different ideas.

- **Method 1 (UniGrad.Correct): Online Ensemble with Injected Corrections**. We propose the UniGrad.Correct algorithm, which consists of a three-layer ensemble structure and incorporates *injected corrections* to ensure stability cancellation within the online ensemble. The algorithm simultaneously achieves regret bounds of $\mathcal{O}(\frac{1}{\lambda} \log V_T)$ for λ-strongly convex functions, $\mathcal{O}(\frac{d}{\alpha} \log V_T)$ for α-exp-concave functions, and $\mathcal{O}(\sqrt{V_T \log V_T})$ for convex functions.

- **Method 2 (UniGrad.Bregman): Online Ensemble with Extracted Bregman Divergence**. We propose the UniGrad.Bregman algorithm, which leverages a negative term from the *extracted Bregman divergence* in linearization and employs a novel analysis that bypasses stability arguments to handle gradient variation. The algorithm simultaneously achieves regret bounds of $\mathcal{O}(\frac{1}{\lambda}\log V_T)$ for λ-strongly convex functions, $\mathcal{O}(\frac{d}{\alpha}\log V_T)$ for α-exp-concave functions, and $\mathcal{O}(\sqrt{V_T})$ for convex functions.

Although UniGrad.Correct exhibits slight suboptimality in the convex case compared to UniGrad.Bregman, the two methods rely on fundamentally different principles and enjoy their own merits. Table 1 provides a comparison of our results with prior works and our conference version (Yan et al., 2023), highlighting that we are the first to achieve gradient-variation regret bounds across all three types of functions simultaneously. Both UniGrad.Correct and UniGrad.Bregman require $\mathcal{O}(\log T)$ gradient queries per round, as they maintain this many base learners in parallel. To improve efficiency, we further develop UniGrad++, which matches the same regret guarantees with only 1 gradient query per round. The key improvement is a careful deployment of the "surrogate optimization" technique, which broadcasts global gradient information to all base learners while effectively handling bias. Additionally, we extend our method to an anytime variant, eliminating the requirement of the time horizon T in advance and preserving the same guarantees.

Technical Contributions. To achieve gradient-variation adaptivity, it is typical to first establish regret guarantees with respect to the *empirical* gradient-variation quantity $\bar{V}_T \triangleq \sum_{t=2}^T \|\nabla f_t(\mathbf{x}_t) - \nabla f_{t-1}(\mathbf{x}_{t-1})\|^2$, and then convert them to the desired V_T-type bounds (see definition in Eq. (1.2)) by handling the additional positive terms. This requires carefully extracting proper positive terms and canceling them by leveraging negative terms in the regret analysis and algorithm design comprehensively, as well as exploiting additional curvature-induced negative terms in the exp-concave and strongly convex cases. Crucially, all these considerations must be compatible with the online ensemble structure, which demands careful design and, in some cases, surgical adjustments across meta-base layers. Below, we discuss key techniques of each method (with further comparisons in Section 7), along with UniGrad++ and the anytime variant.

- **Techniques of UniGrad.Correct.** The key challenge here is to handle the *stability* term $\sum_{t=2}^T \|\mathbf{x}_t - \mathbf{x}_{t-1}\|^2$ in universal online learning. This requires the meta algorithm to achieve an optimistic second-order regret while retaining a stability negative term in the analysis. To this end, we employ a two-layer mirror-descent-based meta algorithm, resulting in an overall three-layer online ensemble structure. We develop a cascaded correction mechanism to cancel stability in this three-layer ensemble and design appropriate optimism to ensure adaptivity across all three function classes. Owing to this explicit stability cancellation, UniGrad.Correct preserves the RVU (Regret bounded by Variations in Utility) property, which is essential for achieving fast convergence in online games (Syrgkanis et al., 2015).

- **Techniques of UniGrad.Bregman.** We employ a fundamentally different approach by conducting a novel smoothness analysis of gradient variations. This enables handling a positive term unrelated to stability, thereby bypassing the need for stability-induced negative terms in the meta algorithm. Combined with a new *Bregman-divergence negative term* extracted from the linearization, the algorithm avoids stability arguments entirely and employs a simple two-layer structure, achieving the optimal $\mathcal{O}(\sqrt{V_T})$ regret for convex functions. This optimal universal rate directly yields optimal guarantees for the stochastically extended adversarial (SEA) model (Chen et al., 2024).

- **Techniques of UniGrad++.** To improve gradient query efficiency, instead of relying on the multi-gradient information $\{\nabla f_t(\mathbf{x}_{t,i})\}_{i=1}^N$, UniGrad++ adopts surrogate optimization by using only the global gradient $\nabla f_t(\mathbf{x}_t)$ to construct different surrogate functions that incorporate curvature information and then feed them into the meta and base updates. Consequently, it is crucial to address the bias introduced by the surrogate function and to handle the additional positive term arising from gradient variations with respect to the surrogate functions.

- **Techniques of Anytime variant.** For the anytime variant, since the algorithm does not know the time horizon T in advance, it is impossible to predefine the number of base learners, which is originally set as $\mathcal{O}(\log T)$. Moreover, the doubling trick cannot be employed in this case, as it would introduce $\mathrm{poly}(\log T)$ regret degradations, thereby ruining the desired gradient-variation bounds for exp-concave and strongly convex functions. We design a dynamic online ensemble framework where the number of base learners is adjusted dynamically based on certain monitoring metrics.

Implication and Applications. We demonstrate the importance and generality of our results through several implications and applications. *(i)* The obtained gradient-variation regret bounds not only safeguard worst-case guarantees (van Erven and Koolen, 2016; Wang et al., 2019) but also directly imply the small-loss bounds of Zhang et al. (2022a) and the gradient-variance bounds of Hazan and Kale (2010) in analysis. *(ii)* Gradient variation is shown to play an essential role in the stochastically extended adversarial (SEA) model (Sachs et al., 2022; Chen et al., 2024), an interpolation between stochastic and adversarial convex optimization. Our approach positively resolves a major open problem left in Chen et al. (2024) on whether it is possible to develop a single algorithm with universal guarantees across strongly convex, exp-concave, and convex functions in the SEA model. *(iii)* In game theory, gradient variation captures changes in other players' actions and facilitates fast convergence to the Nash equilibrium with stability cancellation arguments (Rakhlin and Sridharan, 2013b; Syrgkanis et al., 2015; Zhang et al., 2022b), and we apply UniGrad.Correct to two-player zero-sum games to illustrate its universality.

Comparison to Conference Version. This journal extension significantly improves upon our earlier conference papers (Yan et al., 2023, 2024) in algorithm design, regret analysis, presentation, and experimental evaluation. Specifically, while UniGrad.Correct still employs a three-layer online ensemble to achieve the desired gradient-variation regret, the initial algorithm of Yan et al. (2023) required maintaining $\mathcal{O}((\log T)^2)$ base learners. In contrast, the new design reduces the number of base learners to $\mathcal{O}(\log T)$, substantially improving computational efficiency. This improvement is enabled by a sharper understanding of the three-layer online ensemble, leading to a new construction of correction terms injected into the meta algorithm's feedback loss. More comparisons are discussed in Section 7.2. Additionally, we develop an anytime variant of Yan et al. (2024) that eliminates the need for T in advance, which is achieved through a novel dynamic online ensemble framework that adjusts the number of base learners based on monitoring metrics. Lastly, we have made substantial improvements to the presentation, introducing a more systematic and unified framework for the two methods and a modular structure for technical proofs. We then present the one-gradient version, with the critical role of surrogate loss design highlighted. We also include additional implications and applications to broaden the scope and significance of our methods and conduct empirical evaluation to validate their effectiveness.

Organization. In the following, we first formally state the problem setup and review a general framework for universal online learning in Section 2. Next, we provide the main technical results in Section 3 and Section 4, where two methods with universal gradient-variation regret bounds are developed. Section 5 enhances the efficiency by ensuring only 1 gradient query per round. Then, we discuss the implications, applications, and extensions of the obtained gradient-variation universal regret bounds in Section 6. Based on the technical details and provided applications, Section 7 offers detailed discussions of the two methods and the extension over the conference version. Section 8 reports the experiments. Finally, Section 9 concludes the paper. All proofs and omitted details are deferred to appendices.

2. Problem Setup and Preliminaries

In this section, we introduce preliminaries, including the problem setup, assumptions, the optimistic online mirror descent, and a general universal online learning framework.

Notations. We use $\|\cdot\|$ for $\|\cdot\|_2$ by default. We represent the i-th out of d dimensions of the bold vector $\mathbf{v}$ (or $\boldsymbol{v}$) using the corresponding regular font v_i, i.e., $\mathbf{v}$ (or $\boldsymbol{v}$) $= (v_1, v_2, \ldots, v_d)^\top$. $\|\mathbf{x}\|_U \triangleq \sqrt{\mathbf{x}^\top U \mathbf{x}}$ refers to the matrix norm for any $\mathbf{x}$, where U is a positive semi-definite matrix. For a strictly convex and differentiable function $\psi : \mathcal{X} \to \mathbb{R}$, the induced Bregman divergence is defined as $\mathcal{D}_\psi(\mathbf{x}, \mathbf{y}) \triangleq \psi(\mathbf{x}) - \psi(\mathbf{y}) - \langle \nabla \psi(\mathbf{y}), \mathbf{x} - \mathbf{y} \rangle$. We use Δ_d to represent a d-dimensional simplex and denote the i-th basis vector by $\boldsymbol{e}_i$. We adopt the asymptotic notations $a \lesssim b$ or $a = \mathcal{O}(b)$ to denote that there exists a constant $C < \infty$ such that $a \le Cb$. We use the $\mathcal{O}(\cdot)$-notation to highlight the dependence on T and V_T while treating the iterated logarithmic factors as a constant following previous work (Adamskiy et al., 2012; Luo and Schapire, 2015; Zhao et al., 2024).

2.1 Problem Setup

The protocol of online convex optimization (OCO) is as follows: at each round $t \in [T]$, the learner will select a decision $\mathbf{x}_t \in \mathcal{X} \subseteq \mathbb{R}^d$, while the environment simultaneously chooses a convex function $f_t : \mathcal{X} \to \mathbb{R}$. The learner then incurs a loss $f_t(\mathbf{x}_t)$ and observes the gradient information of the online function $f_t(\cdot)$. Following Zhao et al. (2024), the OCO setting can be further refined based on the type of gradient information accessible to the learner:

(i) **multi-gradient feedback**: the learner can access multiple gradients of the online function, that is, $f_t(\cdot)$ at round $t \in [T]$;

(ii) **one-gradient feedback**: the learner can only access one gradient at the decision point, that is, $\nabla f_t(\mathbf{x}_t)$ at round $t \in [T]$.

We will first address the multi-gradient feedback model (in Section 3 and Section 4) and then improve our results to the more challenging one-gradient feedback model in Section 5.

The goal of the online learner is to minimize the regret measure defined in Eq. (1.1). It is now well-established that the regret rates differ significantly depending on the type of online functions and their curvature coefficients. In fact, there are three main classes of online functions: strongly convex, exponentially concave (abbreviated as exp-concave), and convex functions. The formal definitions are as follows (with convex functions omitted).

Definition 1 (Strong Convexity)**.** A function $f(\cdot)$ is λ-strongly convex if $f(\mathbf{x}) - f(\mathbf{y}) \leq \langle \nabla f(\mathbf{x}), \mathbf{x} - \mathbf{y} \rangle - \frac{\lambda}{2} \cdot \|\mathbf{x} - \mathbf{y}\|^2$ holds for any $\mathbf{x}, \mathbf{y} \in \mathcal{X}$.

Definition 2 (Exp-Concavity)**.** A function $f(\cdot)$ is α-exponentially concave (abbreviated as *exp-concave*),[1] if $f(\mathbf{x}) - f(\mathbf{y}) \leq \langle \nabla f(\mathbf{x}), \mathbf{x} - \mathbf{y} \rangle - \frac{\alpha}{2} \cdot \langle \nabla f(\mathbf{x}), \mathbf{x} - \mathbf{y} \rangle^2$ holds for any $\mathbf{x}, \mathbf{y} \in \mathcal{X}$.

We refer to the $-\frac{\lambda}{2} \cdot \|\mathbf{x} - \mathbf{y}\|^2$ term in λ-strongly convex functions and the $-\frac{\alpha}{2} \cdot \langle \nabla f(\mathbf{x}), \mathbf{x} - \mathbf{y} \rangle^2$ term in α-exp-concave functions as the *curvature-induced negative terms*, which play a crucial role in achieving improved regret bounds compared to convex functions. For the problem-independent regret bounds, it is known that the minimax rates are $\mathcal{O}(\frac{1}{\lambda} \log T)$, $\mathcal{O}(\frac{d}{\alpha} \log T)$, and $\mathcal{O}(\sqrt{T})$ for λ-strongly convex, α-exp-concave, and convex functions, respectively (Ordentlich and Cover, 1998; Abernethy et al., 2008). For the more adaptive gradient-variation regret bounds, it is known that different algorithms can be designed for each class of functions to achieve the corresponding regret bounds: $\mathcal{O}(\frac{1}{\lambda} \log V_T)$ regret for λ-strongly convex functions, $\mathcal{O}(\frac{d}{\alpha} \log V_T)$ regret for α-exp-concave functions, and $\mathcal{O}(\sqrt{V_T})$ regret for convex functions (Chiang et al., 2012; Zhang et al., 2022a).

Universal Online Learning. As can be observed from the above discussions, the curvature information is crucial for the regret rate (no matter for the problem-independent or gradient-variation regret), and thus it is crucial for the online learner to choose the correct algorithm with well-tuned parameters for each class of functions. However, this clearly burdens the learner with the prior knowledge of the function type and the parameter characterizing the curvature, hence prohibiting more applications in practice. Given this background, *universal online learning* aims to design a *single* algorithm that can achieve the optimal regret bound for all three classes of online functions simultaneously.

Mathematically, for a sequence of online functions $\{f_t\}_{t=1}^T$ that may belong to one of the three classes – $\mathcal{F}_{\text{sc}}^\lambda$ (for λ-strongly convex functions), $\mathcal{F}_{\text{ec}}^\alpha$ (for α-exp-concave functions), and $\mathcal{F}_{\text{c}}$ (for convex functions), universal online learning algorithm $\mathcal{A}$ aims to attain the following *universal regret* satisfying:

$$\textsc{Reg}_T(\mathcal{A}, \{f_t\}_{t=1}^T) \lesssim \begin{cases} \textsc{Reg}_T(\mathcal{A}_{\text{sc}}, \mathcal{F}_{\text{sc}}^\lambda), & \text{when } \{f_t\}_{t=1}^T \text{ belongs to } \mathcal{F}_{\text{sc}}^\lambda, \\ \textsc{Reg}_T(\mathcal{A}_{\text{ec}}, \mathcal{F}_{\text{ec}}^\alpha), & \text{when } \{f_t\}_{t=1}^T \text{ belongs to } \mathcal{F}_{\text{ec}}^\alpha, \\ \textsc{Reg}_T(\mathcal{A}_{\text{c}}, \mathcal{F}_{\text{c}}), & \text{when } \{f_t\}_{t=1}^T \text{ belongs to } \mathcal{F}_{\text{c}}, \end{cases} \tag{2.1}$$

where $\mathcal{A}_{\text{sc}}$, $\mathcal{A}_{\text{ec}}$, $\mathcal{A}_{\text{c}}$ are the (optimal) algorithms designed for $\mathcal{F}_{\text{sc}}^\lambda$, $\mathcal{F}_{\text{ec}}^\alpha$, and $\mathcal{F}_{\text{c}}$, respectively. The corresponding regret bounds are denoted as $\textsc{Reg}_T(\mathcal{A}_{\text{sc}}, \mathcal{F}_{\text{sc}}^\lambda)$, $\textsc{Reg}_T(\mathcal{A}_{\text{ec}}, \mathcal{F}_{\text{ec}}^\alpha)$,

1. The formal definition of β-exp-concavity is that $\exp(-\beta f(\cdot))$ is concave. Under Assumptions 1 and 2 (see Section 2.2), β-exp-concavity implies Definition 2 with $\alpha = \frac{1}{2} \cdot \min\{1/(4GD), \beta\}$ (Hazan, 2016, Lemma 4.3). For clarity and simplicity, we adopt Definition 2 as an alternative of exp-concavity.

and $\text{REG}_T(\mathcal{A}_c, \mathcal{F}_c)$. For problem-independent regret, the respective rates are $\mathcal{O}(\frac{1}{\lambda}\log T)$, $\mathcal{O}(\frac{d}{\alpha}\log T)$, and $\mathcal{O}(\sqrt{T})$, as achieved by Zhang et al. (2022a). Furthermore, when adapting to gradient variations, the regret improves to $\mathcal{O}(\frac{1}{\lambda}\log V_T)$, $\mathcal{O}(\frac{d}{\alpha}\log V_T)$, and $\mathcal{O}(\sqrt{V_T})$.

2.2 Assumptions and Optimistic Online Mirror Descent

In this subsection, we first present several standard assumptions commonly used in online convex optimization, and then introduce the algorithmic framework of optimistic online mirror descent (OOMD) (Chiang et al., 2012; Rakhlin and Sridharan, 2013a), which serves not only the foundation of many (adaptive) online learning algorithms, but also the basis of our proposed methods for universal online learning.

Assumption 1 (Domain Boundedness)**.** For any $\mathbf{x}, \mathbf{y} \in \mathcal{X} \subseteq \mathbb{R}^d$, the domain diameter satisfies $\|\mathbf{x} - \mathbf{y}\| \le D$.

Assumption 2 (Gradient Boundedness)**.** For all $t \in [T]$ and any $\mathbf{x} \in \mathcal{X}$, the gradient norm of the online functions is bounded as $\|\nabla f_t(\mathbf{x})\| \le G$.

Assumption 3 (Smoothness)**.** For each $t \in [T]$, the online function $f_t(\cdot)$ is L-smooth, i.e., $\|\nabla f_t(\mathbf{x}) - \nabla f_t(\mathbf{y})\| \le L\|\mathbf{x} - \mathbf{y}\|$ holds for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$.

The domain boundedness and gradient boundedness are standard assumptions for regret minimization in OCO (Shalev-Shwartz, 2012; Hazan, 2016). The smoothness assumption on the online functions is necessary for first-order algorithms to achieve the gradient-variation regret (Chiang et al., 2012). While Assumption 3 requires the smoothness on the entire $\mathbb{R}^d$ space here, this assumption can be relaxed to different degrees for our two proposed methods, which will be specified later.

Optimistic Online Mirror Descent. OOMD applies to the optimistic online learning scenario, where in addition to the standard protocol of OCO, at round $t \in [T]$, the learner also has access to an optimistic estimation of the future loss's gradient $\nabla f_t(\mathbf{x}_t)$ denoted by $M_t \in \mathbb{R}^d$, which is called "optimistic vector" or simply "optimism". Based on this information, OOMD updates in the following way:

$$
\begin{aligned}
\mathbf{x}_t &= \operatorname*{arg\,min}_{\mathbf{x} \in \mathcal{X}} \left\{ \eta_t \langle M_t, \mathbf{x} \rangle + \mathcal{D}_{\psi_t}(\mathbf{x}, \widehat{\mathbf{x}}_t) \right\}, \\
\widehat{\mathbf{x}}_{t+1} &= \operatorname*{arg\,min}_{\mathbf{x} \in \mathcal{X}} \left\{ \eta_t \langle \nabla f_t(\mathbf{x}_t), \mathbf{x} \rangle + \mathcal{D}_{\psi_t}(\mathbf{x}, \widehat{\mathbf{x}}_t) \right\},
\end{aligned}
\tag{2.2}
$$

where $\psi_t(\cdot)$ is a regularizer to be specified, $\eta_t > 0$ is a time-varying step size, $\widehat{\mathbf{x}}_t$ is an internal decision. This framework is highly generic and can recover many existing online learning algorithms through flexible configurations (Zhao et al., 2024). A notable fact is that OOMD can achieve an $\mathcal{O}(\sqrt{A_T})$ adaptive bound for convex functions under standard bounded domain and gradient assumptions, where $A_T \triangleq \sum_{t=1}^T \|\nabla f_t(\mathbf{x}_t) - M_t\|^2$ (Rakhlin and Sridharan, 2013a). Essentially, this design represents how to capture the intrinsic/desired adaptivity in the online learning process: when the optimistic vector M_t accurately predicts the actual gradient $\nabla f_t(\mathbf{x}_t)$, the quantity A_T becomes small, leading to improved regret.

Focusing on the gradient-variation regret and the case of known curvature information, we have the following results. For convex functions, setting the optimism as the last-round gradient (i.e., $M_t = \nabla f_{t-1}(\mathbf{x}_{t-1})$) and the Euclidean regularizer $\psi_t(\mathbf{x}) = \frac{1}{2}\|\mathbf{x}\|_2^2$, OOMD recovers the well-known Optimistic Online Gradient Descent (OOGD) (Chiang et al., 2012):

$$\mathbf{x}_t = \Pi_{\mathcal{X}}\left[\widehat{\mathbf{x}}_t - \eta_t M_t\right], \quad \widehat{\mathbf{x}}_{t+1} = \Pi_{\mathcal{X}}\left[\widehat{\mathbf{x}}_t - \eta_t \nabla f_t(\mathbf{x}_t)\right], \tag{2.3}$$

where $\Pi_{\mathcal{X}}[\mathbf{x}] \triangleq \arg\min_{\mathbf{y}\in\mathcal{X}} \|\mathbf{x}-\mathbf{y}\|_2$ is the Euclidean projection onto the feasible domain $\mathcal{X}$. Under standard assumptions, setting the step size as $\eta_t = \min\{D/\sqrt{1+\bar{V}_{t-1}}, 1/(2L)\}$, where $\bar{V}_t \triangleq \sum_{s=1}^{t} \|\nabla f_s(\mathbf{x}_s) - \nabla f_{s-1}(\mathbf{x}_{s-1})\|^2$, OOGD enjoys an $\mathcal{O}(\sqrt{V_T})$ gradient-variation regret bound, which is provably optimal (Chiang et al., 2012).

For λ-strongly convex functions, using the OOGD algorithm with $M_t = \nabla f_{t-1}(\mathbf{x}_{t-1})$ and $\eta_t = 2/\lambda t$, we can obtain an $\mathcal{O}(\frac{1}{\lambda}\log V_T)$ gradient-variation regret bound (Chiang et al., 2012; Zhang et al., 2022a).

For α-exp-concave functions, setting the optimism as the last-round gradient (i.e., $M_t = \nabla f_{t-1}(\mathbf{x}_{t-1})$) and using the regularizer $\psi_t(\mathbf{x}) = \frac{1}{2}\|\mathbf{x}\|_{U_t}^2$ with $U_t = I + \frac{\alpha G^2}{2} I + \frac{\alpha}{2}\sum_{s=1}^{t-1} \nabla f_s(\mathbf{x}_s)\nabla f_s(\mathbf{x}_s)^\top$, OOMD recovers the Optimistic Online Newton Step (OONS) algorithm (Yang et al., 2014):

$$\mathbf{x}_t = \mathop{\arg\min}_{\mathbf{x}\in\mathcal{X}} \left\|\mathbf{x} - (\widehat{\mathbf{x}}_t - U_t^{-1} M_t)\right\|_{U_t}^2, \quad \widehat{\mathbf{x}}_{t+1} = \mathop{\arg\min}_{\mathbf{x}\in\mathcal{X}} \left\|\mathbf{x} - (\widehat{\mathbf{x}}_t - U_t^{-1}\nabla f_t(\mathbf{x}_t))\right\|_{U_t}^2. \tag{2.4}$$

OONS achieves an $\mathcal{O}(\frac{d}{\alpha}\log V_T)$ gradient-variation regret bound (Yang et al., 2014).

2.3 A General Framework for Universal Online Learning

As presented in Section 2.2, while the same algorithmic template (OOMD) can be used to achieve gradient-variation regret bounds across different function classes, the specific configurations such as step size tuning and regularization are vastly different. This requires the online learner to select the "correct" algorithm and configuration to ensure the favorable guarantees. Universal online learning seeks to eliminate this burden by designing a single algorithm that does not require prior knowledge of the function type or curvature, yet still achieves the same regret bounds as if this information were known.

Now we will review a general framework for universal online learning and the key insight of Zhang et al. (2022a), which achieves the minimax optimal regret bounds of $\mathcal{O}(\frac{1}{\lambda}\log T)$ for λ-strongly convex, $\mathcal{O}(\frac{d}{\alpha}\log T)$ for α-exp-concave, and $\mathcal{O}(\sqrt{T})$ for convex functions. We will also discuss the challenges of adapting this framework to the gradient-variation regret.

Online Ensemble for Universal Online Learning. The fundamental challenge in universal online learning lies in the *uncertainty* of the function type and curvature parameters. A common wisdom is to employ an *online ensemble* with a meta-base two-layer structure, where multiple diverse base learners are deployed to explore the environment and a meta algorithm runs on top to dynamically track the best-performing base learner (van Erven and Koolen, 2016; van Erven et al., 2021; Zhang et al., 2022a; Yan et al., 2023, 2024). Without loss of generality, we can focus on the case where parameters $\alpha, \lambda \in [1/T, 1]$. If $\alpha, \lambda < 1/T$, even the optimal minimax results—$\mathcal{O}(\frac{d}{\alpha}\log T)$ for exp-concave functions and $\mathcal{O}(\frac{1}{\lambda}\log T)$ for strongly convex functions (Hazan et al., 2007)—become linear in T, making the regret bounds vacuous. Conversely, if $\alpha, \lambda > 1$, they can be treated as $\alpha, \lambda = 1$, which only worsens the regret by an ignorable constant factor.

For the non-degenerated case of $\alpha, \lambda \in [1/T, 1]$, we can discretize the unknown α and λ into a candidate pool $\mathcal{H}^{\text{exp}}$ and $\mathcal{H}^{\text{sc}}$ using an exponential grid, defined as

$$\mathcal{H}^{\text{exp}} = \mathcal{H}^{\text{sc}} \triangleq \left\{ \frac{1}{T}, \frac{2}{T}, \frac{2^2}{T}, \cdots, \frac{2^{n-1}}{T} \right\}, \tag{2.5}$$

where $n = \lceil \log_2 T \rceil + 1 = \mathcal{O}(\log T)$ is the number of candidates. It can be proved that the discretized candidate pool $\mathcal{H}^{\text{exp}}$ and $\mathcal{H}^{\text{sc}}$ can approximate the continuous value of α and λ with only constant errors. Based on the pool, it is natural to design three distinct groups of base learners, each tailored to handle different curvature properties:

(i) *strongly convex* base learners $\{\mathcal{B}_i^{\text{sc}}\}_{i \in [N_{\text{sc}}]}$: $|\mathcal{H}^{\text{sc}}| = n$ in total. Each base learner $\mathcal{B}_i$ runs the algorithm for strongly convex functions with a guess $\lambda_i \in \mathcal{H}^{\text{sc}}$ of the true λ;

(ii) *exp-concave* base learners $\{\mathcal{B}_i^{\text{exp}}\}_{i \in [N_{\text{exp}}]}$: $|\mathcal{H}^{\text{exp}}| = n$ in total. Each base learner $\mathcal{B}_i$ runs the algorithm for exp-concave functions with a guess $\alpha_i \in \mathcal{H}^{\text{exp}}$ of the true α;

(iii) *convex* base learners $\mathcal{B}^{\text{c}}$: only 1 base learner running an algorithm for convex functions.

In total, there are $N \triangleq 1 + |\mathcal{H}^{\text{exp}}| + |\mathcal{H}^{\text{sc}}| = 2n + 1 = \mathcal{O}(\log T)$ base learners. The best base learner is the one with the right guess of the curvature type and the closest guess of the curvature coefficient. For example, suppose the online functions are α-exp-concave (while this is unknown to the online learner), then the right guessed coefficient of the best base learner (indexed by $i^\star$) satisfies $\alpha_{i^\star} \le \alpha \le 2\alpha_{i^\star}$.

In addition, there is a meta algorithm running on top of those base learners. At the t-th round, we denote by $\mathbf{x}_{t,i}$ the decision generated by the i-th base learner, for $i \in [N]$. The meta learner will produce the weight vector $\boldsymbol{p}_t = (p_{t,1}, p_{t,2}, \ldots, p_{t,N})^\top \in \Delta_N$ to combine the base learners adaptively. The final decision is formed as $\mathbf{x}_t = \sum_{i=1}^N p_{t,i} \mathbf{x}_{t,i}$.

The key idea of Zhang et al. (2022a). The online ensemble framework offers a general recipe for constructing a universal online learning algorithm, but the specific designs for base learners and, more critically, the meta algorithm remain undefined. The key innovation of Zhang et al. (2022a) lies in the design of the meta algorithm. Their approach starts from the regret decomposition of the two-layer algorithm:

$$\text{Reg}_T = \left[\sum_{t=1}^T f_t(\mathbf{x}_t) - \sum_{t=1}^T f_t(\mathbf{x}_{t,i^\star}) \right] + \left[\sum_{t=1}^T f_t(\mathbf{x}_{t,i^\star}) - \min_{\mathbf{x} \in \mathcal{X}} \sum_{t=1}^T f_t(\mathbf{x}) \right], \tag{2.6}$$

where the *meta regret* (first term) evaluates how well the algorithm tracks the best base learner, and the *base regret* (second term) measures the performance of this base learner. The best base learner is the one that runs the algorithm matching the ground-truth function type with the most accurate guess of the curvature. Zhang et al. (2022a) insightfully observe that ensuring the *second-order regret for the meta algorithm* is pivotal for achieving universality. Specifically, the meta algorithm should satisfy:

$$\sum_{t=1}^T \langle \nabla f_t(\mathbf{x}_t), \mathbf{x}_t - \mathbf{x}_{t,i^\star} \rangle = \sum_{t=1}^T \langle \boldsymbol{p}_t, \boldsymbol{\ell}_t \rangle - \sum_{t=1}^T \ell_{t,i^\star} = \sum_{t=1}^T r_{t,i^\star} \le \mathcal{O}\left(\sqrt{\sum_{t=1}^T r_{t,i^\star}^2} \right) \tag{2.7}$$

where the feedback loss is defined as $\ell_{t,i} \triangleq \langle \nabla f_t(\mathbf{x}_t), \mathbf{x}_{t,i} \rangle$, and hence $r_{t,i} = \langle \nabla f_t(\mathbf{x}_t), \mathbf{x}_t - \mathbf{x}_{t,i} \rangle$ represents the instantaneous regret for the meta algorithm for $i \in [N]$. This condition can be satisfied by advanced prediction-with-expert-advice (PEA) algorithms, such as Adapt-ML-Prod (Gaillard et al., 2014).

By combining this condition with the *curvature-induced negative terms*, Zhang et al. (2022a) demonstrate that the meta regret can be bounded by a *constant* $\mathcal{O}(1)$ for exp-concave and strongly convex functions, while ensuring $\mathcal{O}(\sqrt{T})$ for convex functions. Taking α-exp-concave functions as an example, by definition, the meta regret can be bounded as

$$\text{Meta-Reg} \leq \sum_{t=1}^{T} r_{t,i^\star} - \frac{\alpha}{2} \sum_{t=1}^{T} r_{t,i^\star}^2 \lesssim \sqrt{\sum_{t=1}^{T} r_{t,i^\star}^2} - \frac{\alpha}{2} \sum_{t=1}^{T} r_{t,i^\star}^2 \leq \mathcal{O}(1), \tag{2.8}$$

where the first inequality follows from the property of exp-concave functions (Definition 2), and the second step holds by the second-order regret (2.7) of the meta algorithm. The last inequality is by the AM-GM inequality (Lemma 18). A similar derivation applies to strongly convex functions. Eq. (2.7) illuminates the importance of both the second-order regret of the meta algorithm and the curvature-induced negative terms in universal online learning. Meanwhile, for the convex case, the second-order regret in Eq. (2.7) still ensures an $\mathcal{O}(\sqrt{T})$ meta regret.

Therefore, with the meta algorithm achieving the second-order regret bound, Zhang et al. (2022a) further employ base learners that directly optimize the base regret: using ONS for exp-concave functions leads to an $\mathcal{O}(\frac{d}{\alpha} \log T)$ base regret; using OGD with an appropriate step size for λ-strongly convex functions yields an $\mathcal{O}(\frac{1}{\lambda} \log T)$ base regret; and using OGD with a proper step size for convex functions results in an $\mathcal{O}(\sqrt{T})$ base regret. Combining these base regret bounds with the corresponding meta regret bounds (i.e., $\mathcal{O}(1)$ for exp-concave and strongly convex functions, and $\mathcal{O}(\sqrt{T})$ for convex functions) yields the desired minimax optimal guarantees for universal online learning.

Challenges for Gradient-Variation Regret. The meta regret of Zhang et al. (2022a) is $\mathcal{O}(1)$ for strongly convex and exp-concave functions, and $\mathcal{O}(\sqrt{T})$ for convex functions. As a consequence, for gradient-variation regret, one can choose base learners with gradient-variation bounds (Chiang et al., 2012) to achieve final regret bounds of $\mathcal{O}(\frac{1}{\lambda} \log V_T)$ for λ-strongly convex functions and $\mathcal{O}(\frac{d}{\alpha} \log V_T)$ for α-exp-concave functions. However, for the convex case, since the meta regret is $\mathcal{O}(\sqrt{T})$, it will dominate the final regret even if the base regret can be improved to $\mathcal{O}(\sqrt{V_T})$, resulting in an unfavorable $\mathcal{O}(\sqrt{T})$ overall regret for convex functions that is problem-independent. In the following two sections, we will present novel methods building upon Zhang et al. (2022a) to fix the issue and achieve the desired universal gradient-variation regret across all the three function families.

3. Method I: Online Ensemble with Injected Corrections

This section introduces our first method, UniGrad.Correct, which achieves universal gradient-variation regret bounds for strongly convex, exp-concave, and convex functions.

3.1 Requirement on Meta Algorithm

As discussed in Section 2.3, the main challenge for the existing universal online learning method (Zhang et al., 2022a) in achieving gradient-variation regret is the $\mathcal{O}(\sqrt{T})$ meta regret in the convex case. Therefore, in this subsection, we first analyze the requirements for the meta algorithm and address them in the following subsections.

To achieve adaptivity, we build upon the optimistic online ensemble framework (Zhao et al., 2024), in which it is crucial to introduce the optimistic update in the meta algorithm. Essentially, the meta algorithm is solving the problem of Prediction with Expert Advice (PEA) involving N experts over T rounds. At each round $t \in [T]$, in addition to the feedback loss $\boldsymbol{\ell}_t \in [0,1]^N$ from the environment, optimistic online learning also receives an *optimistic vector* (also called *optimism*), denoted by $\boldsymbol{m}_{t+1} \in \mathbb{R}^N$, that encodes predictable future information. Using this hint, the learner updates the weight vector $\boldsymbol{p}_{t+1} \in \Delta_N$ to minimize cumulative regret: $\sum_{t=1}^T \langle \boldsymbol{\ell}_t, \boldsymbol{p}_t \rangle - \min_{i\in[N]} \sum_{t=1}^T \ell_{t,i}$.

As shown in the analysis surrounding Eqs. (2.7)–(2.8), the *second-order regret* of the meta algorithm is crucial for ensuring universality across different function families. To enjoy gradient-variation adaptivity in the convex case, it is necessary to further incorporate the optimistic update in the meta algorithm. An example of such an algorithm is the Optimistic-Adapt-ML-Prod algorithm (Wei et al., 2016), which can be viewed as an optimistic variant of Adapt-ML-Prod (Gaillard et al., 2014) used in (Zhang et al., 2022a). While we do not present algorithmic details here, we give its *optimistic second-order regret bound* in the following form:

$$\sum_{t=1}^T \langle \boldsymbol{\ell}_t, \boldsymbol{p}_t \rangle - \sum_{t=1}^T \ell_{t,i^\star} \le \mathcal{O}\left(\sqrt{\sum_{t=1}^T \left(r_{t,i^\star} - m_{t,i^\star}\right)^2}\right), \tag{3.1}$$

where $r_{t,i} = \langle \boldsymbol{\ell}_t, \boldsymbol{p}_t \rangle - \ell_{t,i} = \langle \nabla f_t(\mathbf{x}_t), \mathbf{x}_t - \mathbf{x}_{t,i} \rangle$ is the instantaneous regret of the i-th base learner, since we set the feedback loss as $\boldsymbol{\ell}_{t,i} = \langle \nabla f_t(\mathbf{x}_t), \mathbf{x}_{t,i} \rangle$ in the meta update. We are now in a position to design an optimistic vector $\boldsymbol{m}_t \in \mathbb{R}^N$ to attain a favorable meta regret, particularly to avoid the $\mathcal{O}(\sqrt{T})$ meta regret in the convex case.

Indeed, to achieve gradient-variation adaptivity with $V_T \triangleq \sum_{t=2}^T \sup_{\mathbf{x}\in\mathcal{X}} \|\nabla f_t(\mathbf{x}) - \nabla f_{t-1}(\mathbf{x})\|^2$, a common approach in the literature (Chiang et al., 2012; Zhao et al., 2024) is to first obtain an upper bound related to *empirical gradient variations*, defined as $\bar{V}_T \triangleq \sum_{t=2}^T \|\nabla f_t(\mathbf{x}_t) - \nabla f_{t-1}(\mathbf{x}_{t-1})\|^2$, and then account for the additional positive term introduced by the smoothness of the online functions (Assumption 3). In fact, we can relax the assumption to the following one, which only requires the smoothness over the feasible domain $\mathcal{X}$ rather than the entire space $\mathbb{R}^d$.

Assumption 4 (Smoothness over $\mathcal{X}$)**.** For each $t \in [T]$, the online function $f_t(\cdot)$ is L-smooth, i.e., $\|\nabla f_t(\mathbf{x}) - \nabla f_t(\mathbf{y})\| \le L\|\mathbf{x} - \mathbf{y}\|$ holds for any $\mathbf{x}, \mathbf{y} \in \mathcal{X}$.

We then have the following decomposition of the empirical gradient variation.

Lemma 1 (Empirical Gradient Variation Conversion)**.** *Under Assumption 4, the empirical gradient variation can be upper bounded as follows:*

$$\begin{aligned}\bar{V}_T &\le 2\sum_{t=2}^{T}\|\nabla f_t(\mathbf{x}_t)-\nabla f_{t-1}(\mathbf{x}_t)\|^2+2\sum_{t=2}^{T}\|\nabla f_{t-1}(\mathbf{x}_t)-\nabla f_{t-1}(\mathbf{x}_{t-1})\|^2\\ &\le 2\sum_{t=2}^{T}\sup_{\mathbf{x}\in\mathcal{X}}\|\nabla f_t(\mathbf{x})-\nabla f_{t-1}(\mathbf{x})\|^2+2L^2\sum_{t=2}^{T}\|\mathbf{x}_t-\mathbf{x}_{t-1}\|^2.\end{aligned} \tag{3.2}$$

As a result, we can achieve the favorable V_T-type meta regret by eliminating the positive stability term of the final decisions, i.e., $\|\mathbf{x}_t-\mathbf{x}_{t-1}\|^2$. Following the analysis in previous work (Zhao et al., 2024), it can be observed that the final decision $\mathbf{x}_t=\sum_{i=1}^N p_{t,i}\mathbf{x}_{t,i}$ admits a meta-base ensemble update, leading to the following decomposition:

$$\|\mathbf{x}_t-\mathbf{x}_{t-1}\|^2 \lesssim \|\boldsymbol{p}_t-\boldsymbol{p}_{t-1}\|_1^2+\sum_{i=1}^{N}p_{t,i}\|\mathbf{x}_{t,i}-\mathbf{x}_{t-1,i}\|^2, \tag{3.3}$$

where the first part is the meta learner's stability, and the second one is a weighted version of the base learners' stability. The proof is provided in Lemma 10. For now, we focus on the meta stability term, $\|\boldsymbol{p}_t-\boldsymbol{p}_{t-1}\|_1^2$, which typically requires the meta algorithm to contribute a negative regret of the same form to cancel it out, as pioneered in (Zhao et al., 2020) and further developed in the follow-up works (Zhao et al., 2024; Zhang et al., 2022b).

Now the requirements for the meta algorithm are clear: it must not only ensure a regret upper bound with a negative stability term, but also provide a concrete optimism that attains the empirical gradient variation across different function types. Specifically,

(i) **Regret Bound:** The meta algorithm needs to ensure the following *optimistic second-order meta regret* bound with *negative stability terms*:

$$\sum_{t=1}^{T}\langle\boldsymbol{\ell}_t,\boldsymbol{p}_t-\boldsymbol{e}_{i^\star}\rangle \le \mathcal{O}\left(\sqrt{\sum_{t=1}^{T}\left(r_{t,i^\star}-m_{t,i^\star}\right)^2}-\sum_{t=2}^{T}\|\boldsymbol{p}_t-\boldsymbol{p}_{t-1}\|_1^2\right), \tag{3.4}$$

or other similar formulations.

(ii) **Optimism Design:** The meta algorithm needs a concrete and feasible design for the optimism $\boldsymbol{m}_t\in\mathbb{R}^N$ that can effectively unify various function types to achieve the desired $\bar{V}_T$-type (empirical gradient variation) bound.

Based on the above, we briefly clarify our choice of meta algorithm. The meta algorithm should achieve an optimistic second-order regret bound while preserving the negative stability terms as shown in Eq. (3.4). For this purpose, instead of using the Prod-type update like Optimistic-Adapt-ML-Prod, we focus on the mirror-descent-type update, which is well-studied and proven to enjoy negative stability terms in analysis. To the best of our knowledge, the *only* one satisfying both requirements so far is the Multi-scale Multiplicative-weight with Correction (MsMwC) proposed by Chen et al. (2021), which updates as follows:

$$\boldsymbol{p}_t=\operatorname*{arg\,min}_{\boldsymbol{p}\in\Delta_d}\left\{\langle\boldsymbol{m}_t,\boldsymbol{p}\rangle+\mathcal{D}_{\psi_t}(\boldsymbol{p},\widehat{\boldsymbol{p}}_t)\right\},\quad \widehat{\boldsymbol{p}}_{t+1}=\operatorname*{arg\,min}_{\boldsymbol{p}\in\Delta_d}\left\{\langle\boldsymbol{\ell}_t+\boldsymbol{b}_t,\boldsymbol{p}\rangle+\mathcal{D}_{\psi_t}(\boldsymbol{p},\widehat{\boldsymbol{p}}_t)\right\}, \tag{3.5}$$

where $\psi_t(\boldsymbol{p}) = \sum_{i=1}^d \varepsilon_{t,i}^{-1} p_i \log p_i$ is the weighted negative entropy regularizer with time-coordinate-varying learning rate $\varepsilon_{t,i}$, $\boldsymbol{m}_t$ is the optimism, $\boldsymbol{\ell}_t$ is the loss vector, and $\boldsymbol{b}_t$ is a bias term, which is key to solving the "impossible tuning" issue (Chen et al., 2021).

Next, we analyze the negative terms in MsMwC, which are omitted by the authors in their analysis and turn out to be crucial for our purpose. In Lemma 2 below, we extend Lemma 1 of Chen et al. (2021) by explicitly exhibiting the negative terms in MsMwC. The proof is deferred to Appendix A.1.

Lemma 2 (MsMwC Regret)**.** *If* $\max_{t\in[T],i\in[d]}\{|\ell_{t,i}|, |m_{t,i}|\} \le 1$ *and* $\varepsilon_i \le 1/32$*, then* MsMwC *in Eq. (3.5) with time-invariant step sizes (i.e.,* $\varepsilon_{t,i} = \varepsilon_i$ *for any* $t \in [T]$*)*[2] *and bias term* $b_{t,i} = 16\varepsilon_{t,i}(\ell_{t,i} - m_{t,i})^2$ *enjoys:*

$$\sum_{t=1}^T \langle \boldsymbol{\ell}_t, \boldsymbol{p}_t\rangle - \sum_{t=1}^T \ell_{t,i^\star} \le \frac{1}{\varepsilon_{i^\star}} \log \frac{1}{\widehat{p}_{1,i^\star}} + \sum_{i=1}^d \frac{\widehat{p}_{1,i}}{\varepsilon_i} - 8\sum_{t=1}^T\sum_{i=1}^d \varepsilon_i p_{t,i}(\ell_{t,i} - m_{t,i})^2$$
$$+16\varepsilon_{i^\star}\sum_{t=1}^T(\ell_{t,i^\star} - m_{t,i^\star})^2 - 4\sum_{t=2}^T \|\boldsymbol{p}_t - \boldsymbol{p}_{t-1}\|_1^2.$$

With proper step size tuning, Lemma 2 derives an optimistic second-order regret bound with negative stability terms, i.e., $\widetilde{\mathcal{O}}\Big(\sqrt{\sum_{t=1}^T(\ell_{t,i^\star} - m_{t,i^\star})^2} - \sum_{t=2}^T \|\boldsymbol{p}_t - \boldsymbol{p}_{t-1}\|_1^2\Big)$, which is much closer to the desired one in Eq. (3.4), where $\widetilde{\mathcal{O}}(\cdot)$-notation omits $\mathrm{poly}(\log T)$ factors. Nonetheless, there are two caveats.

(i) The second-order bound of MsMwC is based on $(\ell_{t,i^\star} - m_{t,i^\star})^2$, which differs from $(r_{t,i^\star} - m_{t,i^\star})^2$ in Eq. (3.4) and is in fact stronger. To see this, note that the OMD update of Chen et al. (2021) in Eq. (3.5) enjoys a *shifting-invariant property*, meaning that adding a constant to *all* entries of the loss vector does not change the update of $\boldsymbol{p}_t$. Therefore, we can define $\widetilde{m}_{t,i} = \langle \nabla f_t(\mathbf{x}_t), \mathbf{x}_t\rangle - m_{t,i}$ for any $i \in [d]$ as the optimism in MsMwC. With this choice, $(\ell_{t,i^\star} - \widetilde{m}_{t,i^\star})^2 = (r_{t,i^\star} - m_{t,i^\star})^2$, while the update of $\boldsymbol{p}_t$ remains unchanged. In other words, even when using the original optimism $m_{t,i}$, the algorithm still enjoys the second-order regret bound scaling with $(r_{t,i^\star} - m_{t,i^\star})^2$.

(ii) The aforementioned bound of MsMwC omits $\mathrm{poly}(\log T)$ factors. Unfortunately, this makes it infeasible for the strongly convex or exp-concave cases, since the target rate is $\mathcal{O}(\log V_T)$, and an $\widetilde{\mathcal{O}}(1) = \mathcal{O}(\log T)$ meta regret would ruin the desired gradient-variation adaptivity.

In Section 3.2, we design optimism compatible with various function types, where the shift-invariant property plays a key role. We then introduce a new meta algorithm in Section 3.3, which builds on MsMwC and further consists of a two-layer structure to eliminate the additional $\mathcal{O}(\log T)$ factor in the meta regret. Finally, in Section 3.4, we combine these components to present the overall algorithm and its regret guarantees.

3.2 Optimistic Second-Order Meta Regret: A Universal Optimism Design

In the following, we will demonstrate that designing an optimism $m_{t,i}$ to effectively unify various function types with the desired adaptivity is non-trivial, necessitating novel ideas.

2. We only focus on the proof with fixed learning rate, since it is sufficient for our analysis.

A First Attempt on Optimism Design. Examining the optimistic second-order regret bound of the meta algorithm in Eq. (3.1) and the analysis around Eq. (2.8), it is known that the meta regret for base learners associated with exp-concave and strongly convex functions (i.e., $i \in [N_{\text{sc}}]$ and $i \in [N_{\text{exp}}]$, respectively) is bounded by a constant. Therefore, a natural choice for the optimism $\boldsymbol{m}_t \in \mathbb{R}^N$ is:

$$m_{t,i} = \langle \nabla f_{t-1}(\mathbf{x}_{t-1}), \mathbf{x}_{t-1} - \mathbf{x}_{t,i} \rangle \text{ for } i \in [N_{\text{c}}], \text{ and } m_{t,i} = 0 \text{ for } i \in [N_{\text{exp}}] \cup [N_{\text{sc}}].^{3} \quad (3.6)$$

This essentially keeps the optimism for exp-concave and strongly convex base learners to zero, while approximating the instantaneous regret $r_{t,i} = \langle \nabla f_t(\mathbf{x}_t), \mathbf{x}_t - \mathbf{x}_{t,i} \rangle$ for the convex base learner as closely as possible using the last-round decision $\mathbf{x}_{t-1}$ and the latest base decisions $\{\mathbf{x}_{t,i}\}_{i \in [N]}$. However, for the non-zero entries, it becomes challenging to quantify the upper bound of the term $(r_{t,i} - m_{t,i})^2 = (\langle \nabla f_t(\mathbf{x}_t), \mathbf{x}_t - \mathbf{x}_{t,i} \rangle - \langle \nabla f_{t-1}(\mathbf{x}_{t-1}), \mathbf{x}_{t-1} - \mathbf{x}_{t,i} \rangle)^2$ due to a mismatch in indices.

To tackle this challenge, inspired by the literature (Wei et al., 2016; Chen et al., 2021), one possibility is to make the optimism slightly "lookahead", leveraging the shift-invariant property of MsMwC. Specifically, we can set the optimism vector $\boldsymbol{m}_t \in \mathbb{R}^N$ as:

$$m_{t,i} = \langle \nabla f_{t-1}(\mathbf{x}_{t-1}), \mathbf{x}_t - \mathbf{x}_{t,i} \rangle, \quad \forall i \in [N]. \quad (3.7)$$

Although $\mathbf{x}_t$ is unknown when defining $m_{t,i}$, all entries of $\boldsymbol{m}_t$ share the same unknown value $\langle \nabla f_{t-1}(\mathbf{x}_{t-1}), \mathbf{x}_t \rangle$, making it equivalent to using $\widetilde{m}_{t,i} = \langle \nabla f_{t-1}(\mathbf{x}_{t-1}), \mathbf{x}_{t,i} \rangle$ for $i \in [N]$, and the OMD-type update remains unchanged. Under the optimism in Eq. (3.7), the second-order optimistic quantity in the meta regret can be bounded as follows:

$$(r_{t,i^\star} - m_{t,i^\star})^2 \lesssim \begin{cases} \|\mathbf{x}_t - \mathbf{x}_{t,i^\star}\|^2, & \text{(strongly convex)} \\ \langle \nabla f_t(\mathbf{x}_t) - \nabla f_{t-1}(\mathbf{x}_{t-1}), \mathbf{x}_t - \mathbf{x}_{t,i^\star} \rangle^2, & \text{(exp-concave)} \\ \|\nabla f_t(\mathbf{x}_t) - \nabla f_{t-1}(\mathbf{x}_{t-1})\|^2. & \text{(convex)} \end{cases}$$

This works well for the convex case, since it yields a $\bar{V}_T$-type bound that can be converted into the desired V_T bound by addressing the additional positive term later (see Lemma 1). It also works for the strongly convex case, where the upper bound is canceled by the curvature-induced negative term $-\|\mathbf{x}_t - \mathbf{x}_{t,i^\star}\|^2$ from strong convexity (see Definition 1). However, this design (3.7) would *fail* for exp-concave base learners, because the curvature-induced negative term $-\langle \nabla f_t(\mathbf{x}_t), \mathbf{x}_t - \mathbf{x}_{t,i^\star} \rangle^2$ from exp-concavity (see Definition 2) *cannot* cancel the positive term $\langle \nabla f_t(\mathbf{x}_t) - \nabla f_{t-1}(\mathbf{x}_{t-1}), \mathbf{x}_t - \mathbf{x}_{t,i^\star} \rangle^2$ in the meta regret due to a mismatch.

Our Unifying Optimism Design. To unify various types of functions, we propose a simple optimism design: set the optimism as the last-round instantaneous regret, i.e.,

$$m_{t,i} \triangleq r_{t-1,i} = \langle \nabla f_{t-1}(\mathbf{x}_{t-1}), \mathbf{x}_{t-1} - \mathbf{x}_{t,i} \rangle, \quad \forall i \in [N]. \quad (3.8)$$

Unlike the "lookahead" design in (3.7), we simply use the last-round information. The key idea is that, although the resulting optimistic second-order regret bound in (3.1) cannot be

3. In Section 3.2 and Section 3.3, we ignore the requirement of $\max_{t \in [T], i \in [d]} \{|\ell_{t,i}|, |m_{t,i}|\} \le 1$ only for clarity. When presenting the final and formal setups of the losses and optimisms of the meta algorithm in Section 3.4, we will ensure that this requirement is satisfied using normalization.

perfectly canceled by the exp-concavity-induced negative term (i.e., $-r_{t,i^\star}^2$) on each round, it becomes manageable when *aggregated over the entire horizon*:

$$\sum_{t=1}^{T}(r_{t,i^\star} - m_{t,i^\star})^2 \stackrel{(3.8)}{=} \sum_{t=1}^{T}(r_{t,i^\star} - r_{t-1,i^\star})^2 \lesssim 4\sum_{t=1}^{T} r_{t,i^\star}^2. \tag{3.9}$$

This holds because $r_{t,i^\star}$ and $r_{t-1,i^\star}$ differ only by one step, making the cumulative sum easier to control than the individual terms. Lemma 3 shows that this design achieves universality, in particular resolving the failure in the exp-concave case.

Lemma 3 (Universality of Optimism)**.** *Under Assumption 1–3, when setting the optimism as in Eq. (3.8), it holds that*

$$\sum_{t=1}^{T}(r_{t,i^\star} - m_{t,i^\star})^2 \lesssim \begin{cases} \sum_{t=1}^{T} \|\mathbf{x}_t - \mathbf{x}_{t,i^\star}\|^2, & \textit{(strongly convex)} \\ \sum_{t=1}^{T} \langle \nabla f_t(\mathbf{x}_t), \mathbf{x}_t - \mathbf{x}_{t,i^\star}\rangle^2, & \textit{(exp-concave)} \\ \sum_{t=2}^{T} \|\nabla f_t(\mathbf{x}_t) - \nabla f_{t-1}(\mathbf{x}_{t-1})\|^2. & \textit{(convex)} \end{cases}$$

The proof is in Appendix A.2. For strongly convex and exp-concave learners, the meta regret is effectively canceled by curvature-induced negative terms, following the same analysis as in Zhang et al. (2022a) (see Eq. (2.8)). For the convex case, we achieve a $\bar{V}_T$-type bound scaling with the empirical gradient variation. As shown in Lemma 1, this can be further reduced to the desired $\mathcal{O}(\sqrt{V_T})$ meta regret by addressing the extra positive stability term of the final decisions, i.e., $\|\mathbf{x}_t - \mathbf{x}_{t-1}\|^2$, which will be discussed in the next subsection.

3.3 A New Meta Algorithm: Negative Regret Terms and Injected Corrections

In this part, we present the complete meta algorithm design for this problem. To motivate this, we recall that the meta algorithm is required to ensure an optimistic second-order regret bound with negative stability terms as in Eq. (3.4). MsMwC (Chen et al., 2021) is the only known algorithm satisfying both requirements, but its regret bound contains an additional $\mathrm{poly}(\log T)$ factor, making it *infeasible* for strongly convex or exp-concave cases.

To address this issue, we design a new meta algorithm termed MoM (MsMwC-over-MsMwC), which itself is a *two-layer* algorithm using MsMwC as both meta and base learners. The key insight is that the $\log T$ factor arises from the multi-scale regularizer with *time-varying* learning rates. While this regularizer is crucial for resolving the "impossible tuning" issue addressed in their paper, it introduces undesired $\mathrm{poly}(\log T)$ factors due to the clipping issue, which is intolerable for our setting. To overcome this, our proposed MoM meta algorithm uses MsMwC with *time-invariant* learning rates as a base learner, maintaining multiple base learners with different candidate learning rates and dynamically searching for a suitable one to achieve adaptivity. This approach effectively replaces the additional $\mathcal{O}(\log T)$ factors with $\mathcal{O}(\log \sum_t (\ell_{t,i^\star} - m_{t,i^\star})^2)$, a tolerable overhead introduced by the two-layer structure, making it possible to achieve the desired $\mathcal{O}(\log V_T)$ regret for strongly convex and exp-concave cases.

Algorithm 1 MsMwC-over-MsMwC (MoM): Meta algorithm of UniGrad.Correct

Input: Time horizon T, hyperparameter C_0

1: **Initialize**:

- MoM-Top with learning rates $\varepsilon_{t,j}^{\text{TOP}} = \varepsilon_j^{\text{TOP}} = \frac{1}{C_0 \cdot 2^j}$ for all $t \in [T]$ and initial decision $q_{1,j}^{\text{TOP}} = \widehat{q}_{1,j}^{\text{TOP}} = \frac{(\varepsilon_j^{\text{TOP}})^2}{\sum_{j=1}^M (\varepsilon_j^{\text{TOP}})^2}$ for $j \in [M]$
- MoM-Mid with learning rates $\varepsilon_{t,j,i}^{\text{MID}} = 2\varepsilon_j^{\text{TOP}}$ for all $t \in [T]$ and initial decision $q_{1,j,i}^{\text{MID}} = \widehat{q}_{1,j,i}^{\text{MID}} = \frac{1}{N}$ for $i \in [N]$
- Number of MoM-Mid's $M = \lceil \log_2 T \rceil$, number of base learners $N = 2\lceil \log_2 T \rceil + 1$

2: **for** $t = 1$ **to** T **do**

3: Compute the aggregated weight for the next round: $\boldsymbol{p}_t = \sum_{j=1}^M q_{t,j}^{\text{TOP}} \boldsymbol{q}_{t,j}^{\text{MID}} \in \Delta_N$

4: For all $j \in [M]$, the j-th MoM-Mid updates to $\boldsymbol{q}_{t+1,j}^{\text{MID}} \in \Delta_N$ using $b_{t,j,i}^{\text{MID}} = 16\varepsilon_{t,j,i}^{\text{MID}}(\ell_{t,j,i}^{\text{MID}} - m_{t,j,i}^{\text{MID}})^2$ via the following rule:

$$
\begin{aligned}
\widehat{\boldsymbol{q}}_{t+1,j}^{\text{MID}} &= \operatorname*{arg\,min}_{\boldsymbol{q} \in \Delta_N} \left\{ \langle \boldsymbol{\ell}_{t,j}^{\text{MID}} + \boldsymbol{b}_{t,j}^{\text{MID}}, \boldsymbol{q} \rangle + \mathcal{D}_{\psi_{t,j}^{\text{MID}}}(\boldsymbol{q}, \widehat{\boldsymbol{q}}_{t,j}^{\text{MID}}) \right\}, \\
\boldsymbol{q}_{t+1,j}^{\text{MID}} &= \operatorname*{arg\,min}_{\boldsymbol{q} \in \Delta_N} \left\{ \langle \boldsymbol{m}_{t+1,j}^{\text{MID}}, \boldsymbol{q} \rangle + \mathcal{D}_{\psi_{t+1,j}^{\text{MID}}}(\boldsymbol{q}, \widehat{\boldsymbol{q}}_{t+1,j}^{\text{MID}}) \right\}.
\end{aligned} \tag{3.10}
$$

5: MoM-Top updates to $\boldsymbol{q}_{t+1}^{\text{TOP}} \in \Delta_M$ using $b_{t,j}^{\text{TOP}} = 16\varepsilon_{t,j}^{\text{TOP}}(\ell_{t,j}^{\text{TOP}} - m_{t,j}^{\text{TOP}})^2$ via:

$$
\begin{aligned}
\widehat{\boldsymbol{q}}_{t+1}^{\text{TOP}} &= \operatorname*{arg\,min}_{\boldsymbol{q} \in \Delta_M} \left\{ \langle \boldsymbol{\ell}_t^{\text{TOP}} + \boldsymbol{b}_t^{\text{TOP}}, \boldsymbol{q} \rangle + \mathcal{D}_{\psi_t^{\text{TOP}}}(\boldsymbol{q}, \widehat{\boldsymbol{q}}_t^{\text{TOP}}) \right\}, \\
\boldsymbol{q}_{t+1}^{\text{TOP}} &= \operatorname*{arg\,min}_{\boldsymbol{q} \in \Delta_M} \left\{ \langle \boldsymbol{m}_{t+1}^{\text{TOP}}, \boldsymbol{q} \rangle + \mathcal{D}_{\psi_{t+1}^{\text{TOP}}}(\boldsymbol{q}, \widehat{\boldsymbol{q}}_{t+1}^{\text{TOP}}) \right\}.
\end{aligned} \tag{3.11}
$$

6: **end for**

Meta Algorithm. MoM updates in the following way. The first layer runs a single MsMwC (marked as MoM-Top) on Δ_M, whose decision is denoted by $\boldsymbol{q}_t^{\text{TOP}} \in \Delta_M$. It follows the general update rule of (3.5) with its own losses $\{\boldsymbol{\ell}_t^{\text{TOP}}\}_{t=1}^T$, optimisms $\{\boldsymbol{m}_t^{\text{TOP}}\}_{t=1}^T$, bias terms $\{\boldsymbol{b}_t^{\text{TOP}}\}_{t=1}^T$, and learning rates $\{\{\varepsilon_{t,j}^{\text{TOP}}\}_{j=1}^M\}_{t=1}^T$. The weighted negative entropy regularizer ψ_t^{TOP} is defined as $\psi_t^{\text{TOP}}(\boldsymbol{q}) = \sum_{j=1}^M (\varepsilon_{t,j}^{\text{TOP}})^{-1} q_j \log q_j$. MoM-Top further connects with M MsMwCs (marked as MoM-Mid) in the second layer. The decision of the j-th MoM-Mid is denoted by $\boldsymbol{q}_{t,j}^{\text{MID}} \in \Delta_N$, which is updated via the same update rule as in (3.5) with its own losses $\{\boldsymbol{\ell}_{t,j}^{\text{MID}}\}_{t=1}^T$, optimisms $\{\boldsymbol{m}_{t,j}^{\text{MID}}\}_{t=1}^T$, bias terms $\{\boldsymbol{b}_{t,j}^{\text{MID}}\}_{t=1}^T$, and learning rates $\{\{\varepsilon_{t,j,i}^{\text{MID}}\}_{i=1}^N\}_{t=1}^T$. The weighted negative entropy regularizer $\psi_{t,j}^{\text{MID}}$ is defined as $\psi_{t,j}^{\text{MID}}(\boldsymbol{q}) = \sum_{i=1}^N (\varepsilon_{t,j,i}^{\text{MID}})^{-1} q_i \log q_i$. The final output of MoM at the t-th iteration is:

$$
\boldsymbol{p}_t = \sum_{j=1}^M q_{t,j}^{\text{TOP}} \boldsymbol{q}_{t,j}^{\text{MID}} \in \Delta_N. \tag{3.12}
$$

The details of the two-layer meta algorithm MoM are described in Algorithm 1.

Building on Lemma 2, we provide an analysis for the two-layer meta learner MoM, which largely follows Theorems 4 and 5 of Chen et al. (2021), but includes additional negative stability terms. The proof is deferred to Appendix A.3.

Lemma 4 (Two-layer MoM). *If $|\ell_{t,j}^{\text{TOP}}|, |m_{t,j}^{\text{TOP}}|, |\ell_{t,j,i}^{\text{MID}}|, |m_{t,j,i}^{\text{MID}}| \le 1$ and $(\ell_{t,j}^{\text{TOP}} - m_{t,j}^{\text{TOP}})^2 = \langle \boldsymbol{\ell}_{t,j}^{\text{MID}} - \boldsymbol{m}_{t,j}^{\text{MID}}, \boldsymbol{q}_{t,j}^{\text{MID}} \rangle^2$ for any $t \in [T]$, $j \in [M]$, and $i \in [N]$,* MoM *(Algorithm 1) satisfies*

$$\sum_{t=1}^T \langle \boldsymbol{\ell}_t^{\text{TOP}}, \boldsymbol{q}_t^{\text{TOP}} - \boldsymbol{e}_{j^\star} \rangle + \sum_{t=1}^T \langle \boldsymbol{\ell}_{t,j^\star}^{\text{MID}}, \boldsymbol{q}_{t,j^\star}^{\text{MID}} - \boldsymbol{e}_{i^\star} \rangle \le \frac{1}{\varepsilon_{j^\star}^{\text{TOP}}} \log \frac{N}{3C_0^2 (\varepsilon_{j^\star}^{\text{TOP}})^2} + 32 \varepsilon_{j^\star}^{\text{TOP}} V_\star - \frac{C_0}{2} S_T^{\text{TOP}} - \frac{C_0}{4} S_{T,j^\star}^{\text{MID}},$$

where the terms are defined as follows:

- $V_\star \triangleq \sum_{t=2}^T (\ell_{t,j^\star,i^\star}^{\text{MID}} - m_{t,j^\star,i^\star}^{\text{MID}})^2$ *is the second-order quantity;*
- $S_T^{\text{TOP}} \triangleq \sum_{t=2}^T \|\boldsymbol{q}_t^{\text{TOP}} - \boldsymbol{q}_{t-1}^{\text{TOP}}\|_1^2$ *measures the stability of* MoM-Top*;*
- $S_{T,j}^{\text{MID}} \triangleq \sum_{t=2}^T \|\boldsymbol{q}_{t,j}^{\text{MID}} - \boldsymbol{q}_{t-1,j}^{\text{MID}}\|_1^2$ *measures the stability of* MoM-Mid.

We highlight several important points regarding this result. First, following Chen et al. (2021), we choose $M = \mathcal{O}(\log T)$ instances of MoM-Mid to ensure a second-order regret guarantee of $\mathcal{O}(\sqrt{V_\star \log V_\star})$ (Theorem 5 therein). Second, following Lemma 3, we set $m_{t,j^\star,i}^{\text{MID}} = \langle \nabla f_t(\mathbf{x}_t), \mathbf{x}_t \rangle - \langle \nabla f_{t-1}(\mathbf{x}_{t-1}), \mathbf{x}_{t-1} - \mathbf{x}_{t-1,i} \rangle$ to unify various function types. Finally, we note that the condition $(\ell_{t,j}^{\text{TOP}} - m_{t,j}^{\text{TOP}})^2 = \langle \boldsymbol{\ell}_{t,j}^{\text{MID}} - \boldsymbol{m}_{t,j}^{\text{MID}}, \boldsymbol{q}_{t,j}^{\text{MID}} \rangle^2$ in Lemma 4 is only to make the lemma self-contained. When using Lemma 4 (more specifically, in Theorem 3), we will verify that this condition is inherently satisfied by our algorithm.

Injected Corrections. Note that from Lemma 4, the two-layer MoM already consists of negative terms of $\|\boldsymbol{q}_t^{\text{TOP}} - \boldsymbol{q}_{t-1}^{\text{TOP}}\|_1^2$ and $\|\boldsymbol{q}_{t,j}^{\text{MID}} - \boldsymbol{q}_{t-1,j}^{\text{MID}}\|_1^2$. However, observing the decomposition in Eq. (3.3), we can see that those negative terms still *mismatch* with the positive term $\|\boldsymbol{p}_t - \boldsymbol{p}_{t-1}\|_1^2$, where $p_{t,i} = \sum_{j=1}^M q_{t,j}^{\text{TOP}} q_{t,j,i}^{\text{MID}}$ as shown in Eq. (3.12). To solve this issue, we decompose the stability term $\|\boldsymbol{p}_t - \boldsymbol{p}_{t-1}\|_1^2$ into two parts (with proof in Lemma 11):

$$\|\boldsymbol{p}_t - \boldsymbol{p}_{t-1}\|_1^2 \le 2\|\boldsymbol{q}_t^{\text{TOP}} - \boldsymbol{q}_{t-1}^{\text{TOP}}\|_1^2 + 2\sum_{j=1}^M q_{t,j}^{\text{TOP}} \|\boldsymbol{q}_{t,j}^{\text{MID}} - \boldsymbol{q}_{t-1,j}^{\text{MID}}\|_1^2. \tag{3.13}$$

The first term on the right-hand side, $\|\boldsymbol{q}_t^{\text{TOP}} - \boldsymbol{q}_{t-1}^{\text{TOP}}\|_1^2$, can be directly canceled by the corresponding negative term in the analysis of MoM, as shown in Lemma 4. However, the second term, $\sum_{j=1}^M q_{t,j}^{\text{TOP}} \|\boldsymbol{q}_{t,j}^{\text{MID}} - \boldsymbol{q}_{t-1,j}^{\text{MID}}\|_1^2$, cannot be canceled in the same way as the negative term $\|\boldsymbol{q}_{t,j}^{\text{MID}} - \boldsymbol{q}_{t-1,j}^{\text{MID}}\|_1^2$ in Lemma 4 does not align with it. This mismatch presents a key challenge in the analysis. To address this issue, we draw inspiration from the work of Zhao et al. (2024) on the gradient-variation dynamic regret in non-stationary online learning,[4] and introduce carefully designed *correction terms* to facilitate effective collaboration between layers, such that the second term in Eq. (3.13) can be canceled under the universal online learning scenario. This adaptation exhibits more challenges due to the more complicated structure of the employed meta algorithm.

4. This work proposes an improved dynamic regret minimization algorithm compared to its conference version (Zhao et al., 2020), which introduces the correction terms to the meta-base online ensemble structure and thus improves the gradient query complexity from $\mathcal{O}(\log T)$ to 1 within each round.

To see how the correction works, consider a simpler PEA problem with regret $\sum_t \langle \boldsymbol{\ell}_t, \boldsymbol{q}_t - \boldsymbol{e}_{j^\star} \rangle$. If we instead optimize the corrected loss $\boldsymbol{\ell}_t + \boldsymbol{c}_t$ and obtain a regret bound of R_T, i.e., $\sum_t \langle \boldsymbol{\ell}_t + \boldsymbol{c}_t, \boldsymbol{q}_t - \boldsymbol{e}_{j^\star} \rangle \le R_T$, then moving the correction terms to the right-hand side, the original regret is at most $\sum_t \langle \boldsymbol{\ell}_t, \boldsymbol{q}_t - \boldsymbol{e}_{j^\star} \rangle \le R_T - \sum_t \sum_j q_{t,j} c_{t,j} + \sum_t c_{t,j^\star}$, where the *correction-induced negative term* $-\sum_t \sum_j q_{t,j} c_{t,j}$ can be used for cancellation. Meanwhile, the algorithm is required to handle an extra term of $\sum_t c_{t,j^\star}$, which only relies on the $j^\star$-th dimension and is thus relatively easier to control within that dimension (or called expert).

To see how the correction scheme works in our case, we can inject the correction terms into the loss of MoM-Top as:

$$\begin{aligned} \ell_{t,j}^{\text{TOP}} &= \langle \boldsymbol{\ell}_t^{\text{MID}}, \boldsymbol{q}_{t,j}^{\text{MID}} \rangle + \gamma^{\text{TOP}} \| \boldsymbol{q}_{t,j}^{\text{MID}} - \boldsymbol{q}_{t-1,j}^{\text{MID}} \|_1^2, \\ m_{t,j}^{\text{TOP}} &= \langle \boldsymbol{m}_t^{\text{MID}}, \boldsymbol{q}_{t,j}^{\text{MID}} \rangle + \gamma^{\text{TOP}} \| \boldsymbol{q}_{t,j}^{\text{MID}} - \boldsymbol{q}_{t-1,j}^{\text{MID}} \|_1^2, \end{aligned} \tag{3.14}$$

where $\gamma^{\text{TOP}} > 0$ is the coefficient of corrections, which will be specified later. This correction setup is analogous to $c_{t,j} = \gamma^{\text{TOP}} \| \boldsymbol{q}_{t,j}^{\text{MID}} - \boldsymbol{q}_{t-1,j}^{\text{MID}} \|_1^2$ in the simplified example above. As a result, by choosing the correction coefficient appropriately, we can ensure that the correction-induced negative term $-\sum_j q_{t,j}^{\text{TOP}} \| \boldsymbol{q}_{t,j}^{\text{MID}} - \boldsymbol{q}_{t-1,j}^{\text{MID}} \|_1^2$ can be used for cancellation. Moreover, as shown above, the correction introduces a positive term (the cost of corrections) $c_{t,j^\star}$, which equals $\gamma^{\text{TOP}} \| \boldsymbol{q}_{t,j^\star}^{\text{MID}} - \boldsymbol{q}_{t-1,j^\star}^{\text{MID}} \|_1^2$ in our case. Note that this cost of corrections can be perfectly handled by the intrinsic negative terms in the analysis of MoM, as given in Lemma 4. We further note that the construction of the loss and optimism in (3.14) satisfies the requirement $(\ell_{t,j}^{\text{TOP}} - m_{t,j}^{\text{TOP}})^2 = \langle \boldsymbol{\ell}_{t,j}^{\text{MID}} - \boldsymbol{m}_{t,j}^{\text{MID}}, \boldsymbol{q}_{t,j}^{\text{MID}} \rangle^2$ in Lemma 4, which is crucial for the correctness of the regret analysis.

Finally, to conclude, given a PEA problem with regret $\sum_t \langle \boldsymbol{\ell}_t, \boldsymbol{p}_t - \boldsymbol{e}_{i^\star} \rangle$, by leveraging MoM (Algorithm 1) along with corrected losses in Eq. (3.14), it holds that

$$\begin{aligned} &\sum_{t=1}^T \langle \boldsymbol{\ell}_t, \boldsymbol{p}_t - \boldsymbol{e}_{i^\star} \rangle = \sum_{t=1}^T \langle \boldsymbol{\ell}_t, \boldsymbol{p}_t - \boldsymbol{q}_{t,j^\star}^{\text{MID}} \rangle + \sum_{t=1}^T \langle \boldsymbol{\ell}_t, \boldsymbol{q}_{t,j^\star}^{\text{MID}} - \boldsymbol{e}_{i^\star} \rangle \\ &= \sum_{t=1}^T \langle \boldsymbol{\ell}_t^{\text{TOP}}, \boldsymbol{q}_t^{\text{TOP}} - \boldsymbol{e}_{j^\star} \rangle + \sum_{t=1}^T \langle \boldsymbol{\ell}_t^{\text{MID}}, \boldsymbol{q}_{t,j^\star}^{\text{MID}} - \boldsymbol{e}_{i^\star} \rangle - \gamma^{\text{TOP}} \sum_{t=1}^T \sum_{j=1}^M q_{t,j}^{\text{TOP}} \| \boldsymbol{q}_{t,j}^{\text{MID}} - \boldsymbol{q}_{t-1,j}^{\text{MID}} \|_1^2 + \gamma^{\text{TOP}} S_{T,j^\star}^{\text{MID}} \\ &\le \mathcal{O}\left(\sqrt{V_\star \log V_\star} - \sum_{t=2}^T \| \boldsymbol{q}_t^{\text{TOP}} - \boldsymbol{q}_{t-1}^{\text{TOP}} \|_1^2 - \gamma^{\text{TOP}} \sum_{t=1}^T \sum_{j=1}^M q_{t,j}^{\text{TOP}} \| \boldsymbol{q}_{t,j}^{\text{MID}} - \boldsymbol{q}_{t-1,j}^{\text{MID}} \|_1^2 \right) \\ &\le \mathcal{O}\left(\sqrt{V_\star \log V_\star} - \sum_{t=1}^T \| \boldsymbol{p}_t - \boldsymbol{p}_{t-1} \|_1^2 \right), \end{aligned}$$

where $V_\star \triangleq \sum_{t=2}^T (\ell_{t,j^\star,i^\star}^{\text{MID}} - m_{t,j^\star,i^\star}^{\text{MID}})^2$ is the second-order quantity, the second step sets $\boldsymbol{\ell}_t^{\text{MID}} = \boldsymbol{\ell}_t$ and uses the definition of correction terms in Eq. (3.14), the third step leverages Lemma 4, and the last step is due to Eq. (3.13). This nearly matches our goal in Eq. (3.4), up to a logarithmic regret overhead in the second-order optimistic term.

3.4 Overall Algorithm and Regret Guarantee

The meta algorithm proposed in Section 3.3 is able to achieve a second-order regret bound of $\mathcal{O}(\sqrt{V_\star \log V_\star})$ with negative stability terms of $\| \boldsymbol{p}_t - \boldsymbol{p}_{t-1} \|_1^2$. Recall that we still need to

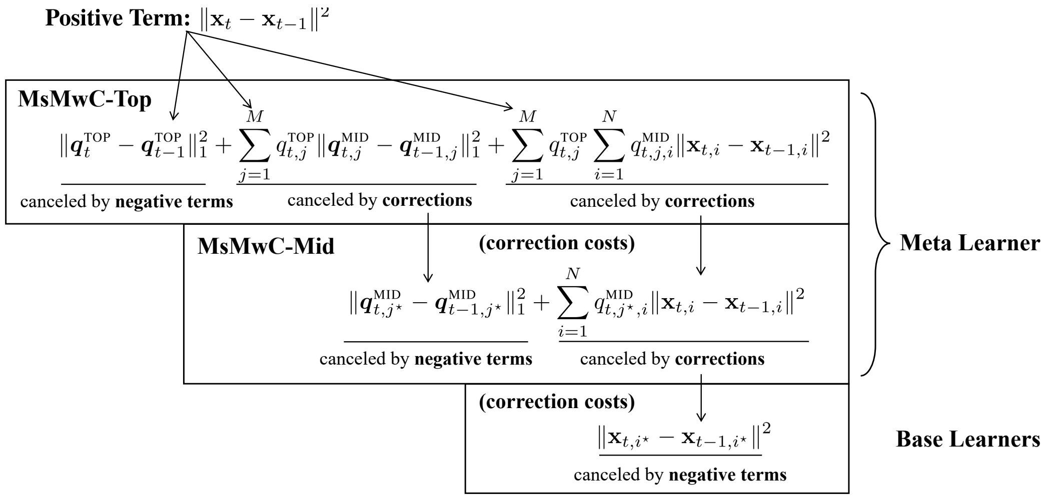


Figure 1: Decomposition of the positive term $\|\mathbf{x}_t - \mathbf{x}_{t-1}\|^2$ and how it is handled by our online ensemble method via intrinsic negative stability terms and injected corrections.

handle $\sum_{i=1}^N p_{t,i}\|\mathbf{x}_{t,i} - \mathbf{x}_{t-1,i}\|^2$ as shown in the second term of Eq. (3.3). To this end, we provide a further decomposition of this quantity:

$$\begin{aligned}\sum_{i=1}^{N} p_{t,i}\|\mathbf{x}_{t,i} - \mathbf{x}_{t-1,i}\|^2 &= \sum_{i=1}^{N}\left(\sum_{j=1}^{M} q_{t,j}^{\text{TOP}} q_{t,j,i}^{\text{MID}}\right)\|\mathbf{x}_{t,i} - \mathbf{x}_{t-1,i}\|^2 \\ &= \sum_{i=1}^{N}\sum_{j=1}^{M} q_{t,j}^{\text{TOP}} q_{t,j,i}^{\text{MID}}\|\mathbf{x}_{t,i} - \mathbf{x}_{t-1,i}\|^2 = \sum_{j=1}^{M} q_{t,j}^{\text{TOP}}\sum_{i=1}^{N} q_{t,j,i}^{\text{MID}}\|\mathbf{x}_{t,i} - \mathbf{x}_{t-1,i}\|^2,\end{aligned} \tag{3.15}$$

where the first equality exploits the two-layer structure of the meta algorithm for computing $p_{t,i}$ in Eq. (3.12). Thus, we obtain the following decomposition of the overall algorithmic stability $\|\mathbf{x}_t - \mathbf{x}_{t-1}\|^2$.

Lemma 5. *For any $t \geq 2$, if $\mathbf{x}_t = \sum_{i=1}^N p_{t,i}\mathbf{x}_{t,i} \in \mathcal{X}$ and $\boldsymbol{p}_t = \sum_{j=1}^M q_{t,j}^{\text{TOP}}\boldsymbol{q}_{t,j}^{\text{MID}} \in \Delta_N$, where $\boldsymbol{q}_t^{\text{TOP}} \in \Delta_M$ and $\boldsymbol{q}_{t,j}^{\text{MID}} \in \Delta_N$ for any $j \in [M]$, then it holds that*

$$\|\mathbf{x}_t - \mathbf{x}_{t-1}\|^2 \leq 4D^2\|\boldsymbol{q}_t^{\text{TOP}} - \boldsymbol{q}_{t-1}^{\text{TOP}}\|_1^2 + 4D^2\sum_{j=1}^{M} q_{t,j}^{\text{TOP}}\|\boldsymbol{q}_{t,j}^{\text{MID}} - \boldsymbol{q}_{t-1,j}^{\text{MID}}\|_1^2 + 2\sum_{j=1}^{M} q_{t,j}^{\text{TOP}}\sum_{i=1}^{N} q_{t,j,i}^{\text{MID}}\|\mathbf{x}_{t,i} - \mathbf{x}_{t-1,i}\|^2.$$

The proof can be directly derived by combining Lemma 10, Eq. (3.13), and Eq. (3.15). It is worth noting that this decomposition differs from the one presented in our conference version (Yan et al., 2023). The specific differences and key improvements will be discussed at the end of this section.

Based on Lemma 5, we leverage the idea of *cancellation by corrections* (Zhao et al., 2024), as already employed in the last subsection. Generally, to handle weighted terms like $\sum_j q_{t,j}^{\text{TOP}} c_{t,j}$, we inject correction terms into the loss of MoM-Top as $\ell_{t,j}^{\text{TOP}} \leftarrow \ell_{t,j}^{\text{TOP}} + c_{t,j}$. Here the correction term $c_{t,j}$ is designed with two components:

$$c_{t,j} \approx \|\boldsymbol{q}_{t,j}^{\text{MID}} - \boldsymbol{q}_{t-1,j}^{\text{MID}}\|_1^2 + \sum_{i=1}^{N} q_{t,j,i}^{\text{MID}}\|\mathbf{x}_{t,i} - \mathbf{x}_{t-1,i}\|^2,$$

allowing us to cancel both the second and third terms in Lemma 5 simultaneously, rather than relying on a single correction as in Eq. (3.14). Thus, the final feedback loss and optimism of MoM-Top are set using two corrections:

$$\ell_{t,j}^{\text{TOP}} = \frac{1}{Z}\left(\langle \boldsymbol{\ell}_t^{\text{MID}}, \boldsymbol{q}_{t,j}^{\text{MID}}\rangle + \gamma^{\text{TOP}}\|\boldsymbol{q}_{t,j}^{\text{MID}} - \boldsymbol{q}_{t-1,j}^{\text{MID}}\|_1^2 + \sum_{i=1}^N \gamma^{\text{MID}} q_{t,j,i}^{\text{MID}}\|\mathbf{x}_{t,i} - \mathbf{x}_{t-1,i}\|^2\right) \in [-1,1],$$
$$m_{t,j}^{\text{TOP}} = \frac{1}{Z}\left(\langle \boldsymbol{m}_t^{\text{MID}}, \boldsymbol{q}_{t,j}^{\text{MID}}\rangle + \gamma^{\text{TOP}}\|\boldsymbol{q}_{t,j}^{\text{MID}} - \boldsymbol{q}_{t-1,j}^{\text{MID}}\|_1^2 + \sum_{i=1}^N \gamma^{\text{MID}} q_{t,j,i}^{\text{MID}}\|\mathbf{x}_{t,i} - \mathbf{x}_{t-1,i}\|^2\right) \in [-1,1],$$
(3.16)

where $\gamma^{\text{TOP}}, \gamma^{\text{MID}} > 0$ are the correction coefficients and $Z > 0$ is a to-be-determined normalization factor to ensure that both the feedback loss and the optimism in Eq. (3.16) lie in $[-1, 1]$. Note that when injecting a correction into the loss, the same correction must be applied to the optimism. This is because the second-order bound depends on the difference between the loss and the optimism, i.e., $\ell_{t,j}^{\text{TOP}} - m_{t,j}^{\text{TOP}}$. Therefore, this preserves the regret guarantees while incorporating corrections.

Recall that the correction scheme comes with a cost. Specifically, when we use the corrections in Eq. (3.16), we need to handle the extra term of $\gamma^{\text{MID}} \sum_{i=1}^N q_{t,j^\star,i}^{\text{MID}} \|\mathbf{x}_{t,i} - \mathbf{x}_{t-1,i}\|^2$ for some specific but unknown $j^\star \in [M]$ in the analysis. To this end, we inject the correction term into the loss of every MoM-Mid as $\ell_{t,j,i}^{\text{MID}} \leftarrow \ell_{t,j,i}^{\text{MID}} + c_{t,i}$, where $c_{t,i} = \gamma^{\text{MID}}\|\mathbf{x}_{t,i} - \mathbf{x}_{t-1,i}\|^2$. To conclude, the final losses and optimisms of MoM-Mid are:

$$\begin{aligned}\ell_{t,j,i}^{\text{MID}} &= \frac{1}{Z}\left(\langle \nabla f_t(\mathbf{x}_t), \mathbf{x}_{t,i}\rangle + \gamma^{\text{MID}}\|\mathbf{x}_{t,i} - \mathbf{x}_{t-1,i}\|^2\right) \in [-1,1],\\ m_{t,j,i}^{\text{MID}} &= \frac{1}{Z}\left(\langle \nabla f_{t-1}(\mathbf{x}_{t-1}), \mathbf{x}_{t-1,i}\rangle + \gamma^{\text{MID}}\|\mathbf{x}_{t,i} - \mathbf{x}_{t-1,i}\|^2\right) \in [-1,1],\end{aligned} \tag{3.17}$$

where Z is set as $\max\{Z_{\text{MID}}, Z_{\text{TOP}}\}$. Specifically, $Z_{\text{MID}} \triangleq GD + \gamma^{\text{MID}} D^2$ serves as the normalization factor ensuring that $\ell_{t,j,i}^{\text{MID}}, m_{t,j,i}^{\text{MID}} \in [-1, 1]$ for all $t \in [T], j \in [M], i \in [N]$, and $Z_{\text{TOP}} \triangleq 1+\gamma^{\text{MID}} D^2+2\gamma^{\text{TOP}}$ is chosen to restrict the range of $\ell_{t,j}^{\text{TOP}}, m_{t,j}^{\text{TOP}}$ for all $t \in [T], j \in [M]$. We note that setting the same normalization factor for MoM-Top and MoM-Mid is necessary to directly apply Lemma 4.

We now present our final meta algorithm MoM (Algorithm 1) with carefully designed injected corrections into both MoM-Top and MoM-Mid. We emphasize that the cost of the corrections in MoM-Mid is a positive term of $\gamma^{\text{MID}}\|\mathbf{x}_{t,i^\star} - \mathbf{x}_{t-1,i^\star}\|^2$ for some specific but unknown $i^\star \in [N]$ in the analysis. Fortunately, this is exactly the stability term of the i-th base learner, which can be perfectly handled by the intrinsic negative terms in the analysis of classic online mirror descent algorithms. We summarize the overall correction process in Figure 1. Because (3.16) and (3.17) both include correction steps, the mechanism exhibits a cascade in which higher-layer corrections propagate to lower layers to cancel negative terms. Thus, we refer to this as a *cascaded correction* mechanism.

To conclude, because we choose the two-layer MoM as the meta learner, the overall algorithm results in a *three-layer* online ensemble structure. The overall update of UniGrad.Correct is presented in Algorithm 2. The base learner configurations are the same as those introduced in Section 2.3. Specifically, in Line 3, the learner submits the weighted decision, suffers the corresponding loss, and receives the gradient information of the loss

Algorithm 2 UniGrad.Correct: Universal Gradient-variation Regret by Injected Corrections

Input: Base learner configurations $\{\mathcal{B}_i\}_{i\in[N]} \triangleq \{\mathcal{B}_i^{\text{sc}}\}_{i\in[N_{\text{sc}}]} \cup \{\mathcal{B}_i^{\exp}\}_{i\in[N_{\exp}]} \cup \mathcal{B}^{\text{c}}$, algorithm parameters γ^{MID}, γ^{TOP}, and C_0

1: **Initialize**: $\mathcal{M}$ — meta algorithm MoM as shown in Algorithm 1
$\{\mathcal{B}_i\}_{i\in[N]}$ — base learners as specified in Section 2.3
2: **for** $t = 1$ **to** T **do**
3: Submit $\mathbf{x}_t = \sum_{i=1}^N p_{t,i}\mathbf{x}_{t,i}$, suffer $f_t(\mathbf{x}_t)$, and observe $\nabla f_t(\cdot)$
4: $\{\mathcal{B}_i\}_{i=1}^N$ update their own decisions to $\{\mathbf{x}_{t+1,i}\}_{i=1}^N$ using $\nabla f_t(\cdot)$
5: Compute $\{\boldsymbol{\ell}_{t,j}^{\text{MID}}, \boldsymbol{m}_{t+1,j}^{\text{MID}}\}_{j=1}^M$ via Eq. (3.17), send to $\mathcal{M}$, get $\{\boldsymbol{q}_{t+1,j}^{\text{MID}}\}_{j=1}^M \in (\Delta_N)^M$
6: Compute $\boldsymbol{\ell}_t^{\text{TOP}}, \boldsymbol{m}_{t+1}^{\text{TOP}}$ via Eq. (3.16), send to $\mathcal{M}$, and obtain $\boldsymbol{q}_{t+1}^{\text{TOP}} \in \Delta_M$
7: Aggregate the final meta weights $\boldsymbol{p}_{t+1} \in \Delta_N$ via Eq. (3.12)
8: **end for**

function. Subsequently, the update is conducted from the bottom to the top. Concretely, in Line 4, the base learners update their own decisions to $\{\mathbf{x}_{t+1,i}\}_{i=1}^N$ using $\nabla f_t(\cdot)$. In Line 5, each MoM-Mid computes its own losses and optimisms using Eq. (3.17) and updates its own decisions according to Algorithm 1. In Line 6, MoM-Top computes its loss and optimism using Eq. (3.16) and updates accordingly. Finally, in Line 7, the meta learner aggregates the final weights $\boldsymbol{p}_{t+1} \in \Delta_N$ via Eq. (3.12).

We next present the main theoretical result. Our proposed UniGrad.Correct can achieve the following gradient-variation regret guarantees, with the proof provided in Appendix A.4.

Theorem 1. *Under Assumptions 1, 2, 4, by setting* $C_0 = \max\{1, 8D, 4\gamma^{\text{TOP}}, 4D^2C_1\}$, $\gamma^{\text{TOP}} = C_1$, *and* $\gamma^{\text{MID}} = 2D^2C_1$, *where* $C_1 = 128(D^2L^2 + G^2)$, UniGrad.Correct *(Algorithm 2) achieves the following universal gradient-variation regret guarantees:*

$$\sum_{t=1}^T f_t(\mathbf{x}_t) - \min_{\mathbf{x}\in\mathcal{X}} \sum_{t=1}^T f_t(\mathbf{x}) \le \begin{cases} \mathcal{O}\left(\frac{1}{\lambda}\log V_T\right), & \text{when } \{f_t\}_{t=1}^T \text{ are } \lambda\text{-strongly convex}, \\ \mathcal{O}\left(\frac{d}{\alpha}\log V_T\right), & \text{when } \{f_t\}_{t=1}^T \text{ are } \alpha\text{-exp-concave}, \\ \mathcal{O}(\sqrt{V_T \log V_T}), & \text{when } \{f_t\}_{t=1}^T \text{ are convex}, \end{cases} \tag{3.18}$$

We emphasize that the regret guarantee is achieved *without* prior knowledge of the function families or curvature information. For strongly convex functions and exp-concave functions, the regret bound matches the best-known gradient-variation regret bounds that were specifically designed with curvature prior knowledge. For convex functions, the result exhibits a slight logarithmic gap compared to the target $\mathcal{O}(\sqrt{V_T})$ bound, and this gap will be addressed and closed in the next section.

It is worth mentioning that even when the curvature α (or λ) is smaller than $1/T$, our algorithm can still guarantee an $\mathcal{O}(\sqrt{V_T \log V_T})$ bound, because exp-concave and strongly convex functions are also convex and thus our convex bound still holds.

Remark 1 (Technique)**.** Our method follows the general *optimistic online ensemble* framework proposed by Zhao et al. (2024), which was originally designed for dynamic regret minimization with convex functions. In the universal online learning with gradient-variation setting, new ingredients are required. Specifically, the goal of handling different function families universally necessitates a two-layer meta algorithm with both second-order regret and negative stability terms, namely, MoM. Furthermore, solving this problem requires new ideas on universal optimism design, the correction scheme, and base-learner sharing, which collectively lead to our three-layer collaborative online ensemble structure, making it significantly different from the two-layer structure used in Zhao et al. (2024). ◁

Remark 2 (Comparison to Conference Version)**.** We here briefly mention the differences between UniGrad.Correct and the conference version (Yan et al., 2023). In the conference version, we have introduced the idea of employing a three-layer online ensemble (with a two-layer MoM as the meta algorithm and cascaded correction terms to address positive stability terms) to achieve gradient-variation universal regret. However, the current UniGrad.Correct differs significantly from the conference version—the conference version requires maintaining $\mathcal{O}((\log T)^2)$ base learners that are updated simultaneously, whereas UniGrad.Correct reduces this complexity to $\mathcal{O}(\log T)$, offering a more efficient solution. Detailed technical comparisons and the improvement will be discussed in Section 7.2. ◁

4. Method II: Online Ensemble with Extracted Bregman Divergence

This section introduces our second method, named UniGrad.Bregman, which achieves optimal universal gradient-variation regret bounds. The new method exhibits a significantly different methodology from the UniGrad.Correct method presented in Section 3.

4.1 Online Ensemble with Extracted Bregman Divergence

This subsection leverages the Bregman divergence term extracted from the linearization of convex online functions, along with a key property of smooth functions that connects gradient variation to Bregman divergence.

For clarity, we illustrate our idea from the ground level. To obtain the gradient variation V_T defined in (1.2), we first need to attain its *empirical* version $\bar{V}_T \triangleq \sum_{t \le T} \|\nabla f_t(\mathbf{x}_t) - \nabla f_{t-1}(\mathbf{x}_{t-1})\|^2$. Previous studies decompose this term as shown in Eq. (3.2), which requires controlling the algorithmic stability $\|\mathbf{x}_t - \mathbf{x}_{t-1}\|^2$. Consequently, since each decision is a weighted combination of base learners' decisions (i.e., $\mathbf{x}_t = \sum_{i \le N} p_{t,i}\mathbf{x}_{t,i}$), the stability is difficult to control. This requires a very powerful meta algorithm with both second-order regret guarantees and negative stability terms. This is why UniGrad.Correct requires a three-layer online ensemble structure and achieves a sub-optimal $\mathcal{O}(\sqrt{V_T \log V_T})$ regret bound for convex functions.

Empirical Gradient Variation Decomposition. To address the above problem, we propose a novel decomposition of the empirical gradient variation that avoids the stability term at the meta algorithm level. Specifically, we decompose the empirical gradient variation into three parts:

$$\bar{V}_T \lesssim \|\nabla f_t(\mathbf{x}_t) - \nabla f_t(\mathbf{x}_{t,i^\star})\|^2 + \|\nabla f_t(\mathbf{x}_{t,i^\star}) - \nabla f_{t-1}(\mathbf{x}_{t-1,i^\star})\|^2 + \|\nabla f_{t-1}(\mathbf{x}_{t-1,i^\star}) - \nabla f_{t-1}(\mathbf{x}_{t-1})\|^2. \tag{4.1}$$

- The middle term on the right-hand side measures the empirical gradient variation of a base learner, which can be controlled within the base learner via its intrinsic negative stability terms using optimistic OMD algorithms.

- The first and last terms capture the gradient differences between the meta decision $\mathbf{x}_t$ (or $\mathbf{x}_{t-1}$) and the best base learner's decision $\mathbf{x}_{t,i^\star}$ (or $\mathbf{x}_{t-1,i^\star}$). We show that they can be upper-bounded by *Bregman divergences* using a useful property of smoothness (Nesterov, 2018) (restated below), which yields $\|\nabla f_t(\mathbf{x}_t) - \nabla f_t(\mathbf{x}_{t,i^\star})\|^2 \le 2L \cdot \mathcal{D}_{f_t}(\mathbf{x}_{t,i^\star}, \mathbf{x}_t)$.

Proposition 1 (Theorem 2.1.5 of Nesterov (2018))**.** *A function $f(\cdot)$ is L-smooth over $\mathbb{R}^d$ if and only if*

$$\|\nabla f(\mathbf{x}) - \nabla f(\mathbf{y})\|^2 \le 2L \cdot \mathcal{D}_f(\mathbf{y}, \mathbf{x}), \quad \textit{for any } \mathbf{x}, \mathbf{y} \in \mathbb{R}^d. \tag{4.2}$$

Compared with the commonly used inequality $\|\nabla f(\mathbf{x}) - \nabla f(\mathbf{y})\| \le L\|\mathbf{x} - \mathbf{y}\|$, Proposition 1 provides a tighter bound for the *squared* gradient changes. Since $\|\nabla f(\mathbf{x}) - \nabla f(\mathbf{y})\|^2 \le 2L\mathcal{D}_f(\mathbf{y}, \mathbf{x}) \le L^2\|\mathbf{x} - \mathbf{y}\|^2$, where the second inequality holds because $\mathcal{D}_f(\mathbf{y}, \mathbf{x}) \le \frac{L}{2}\|\mathbf{x} - \mathbf{y}\|^2$ for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$ (Nesterov, 2018, Theorem 2.1.5), this yields a simpler subsequent analysis.

Combining the above decomposition with the smoothness property yields the following result, which upper-bounds the empirical gradient variation $\bar{V}_T$ by the Bregman divergence terms and the gradient variation V_T. The proof is deferred to Appendix B.1.

Lemma 6 (Empirical Gradient Variation Conversion - II)**.** *Under Assumption 3, the empirical gradient variation $\bar{V}_T \triangleq \sum_{t=2}^T \|\nabla f_t(\mathbf{x}_t) - \nabla f_{t-1}(\mathbf{x}_{t-1})\|^2$ can be upper bounded as*

$$\bar{V}_T \lesssim 2L\sum_{t=1}^T \mathcal{D}_{f_t}(\mathbf{x}_{t,i^\star}, \mathbf{x}_t) + L^2\sum_{t=2}^T \|\mathbf{x}_{t,i^\star} - \mathbf{x}_{t-1,i^\star}\|^2 + V_T, \tag{4.3}$$

where $i^\star \in [N]$ can be any base learner index.

Negative Bregman Divergence. In Lemma 6, the only term that remains to be handled is the Bregman divergence $\mathcal{D}_{f_t}(\mathbf{x}_{t,i^\star}, \mathbf{x}_t)$. Fortunately, this term is canceled by a negative term arising from the linearization of convex functions. Specifically, the meta regret can be transformed into

$$\textsc{Meta-Reg} = \sum_{t=1}^T f_t(\mathbf{x}_t) - \sum_{t=1}^T f_t(\mathbf{x}_{t,i^\star}) = \sum_{t=1}^T \langle \nabla f_t(\mathbf{x}_t), \mathbf{x}_t - \mathbf{x}_{t,i^\star}\rangle - \sum_{t=1}^T \mathcal{D}_{f_t}(\mathbf{x}_{t,i^\star}, \mathbf{x}_t), \tag{4.4}$$

where $i^\star$ indicates the best base learner. The last term is a *negative term from linearization*, which represents the compensation when treating a convex function as linear. Previous studies on gradient-variation regret omit this term, while we show below that this negative term helps achieve a simpler analysis of the empirical gradient variation.

The Bregman divergence negative term in Eq. (4.4) cancels the positive term in Eq. (4.3), and the stability term $\|\mathbf{x}_{t,i^\star} - \mathbf{x}_{t-1,i^\star}\|^2$ can be controlled within the base learner via its intrinsic negative stability terms using optimistic algorithms. Thus, only the gradient variation quantity V_T remains as the desired regret bound.

We emphasize that the Bregman divergence negative term arises from the linearization of convex functions and is thus *algorithm-independent*. Therefore, we avoid controlling the algorithmic stability for gradient variation regret, in contrast to previous works (Chiang et al., 2012; Yan et al., 2023). To the best of our knowledge, this is the *first* alternative analysis of gradient-variation regret since its introduction.

The inspiration for our negative term from linearization comes from recent advances in stochastic smooth optimization (Joulani et al., 2020). While their work focuses on achieving the $\mathcal{O}(1/T^2)$ function-value convergence rate of Nesterov's accelerated gradient method (Nesterov, 2018), our approach addresses the gradient-variation regret in the universal online (adversarial) convex optimization setting.

About Smoothness Requirement. Finally, we discuss the smoothness requirement. Notice that Proposition 1 requires smoothness over the whole $\mathbb{R}^d$, which is a much too strong assumption. We emphasize that to leverage the useful smoothness property in Proposition 1, we only need $\|\nabla f(\mathbf{x}) - \nabla f(\mathbf{y})\|^2 \le 2L\mathcal{D}_f(\mathbf{y}, \mathbf{x})$ on the feasible domain $\mathcal{X}$. In this work, we show that this requirement can be satisfied with smoothness over a slightly larger domain than $\mathcal{X}$, formally, the minimal Assumption 5 below. Readers can refer to Lemma 7 in Appendix B.2 for the formal statement and the proof.

Assumption 5 (Smoothness over $\mathcal{X}_+$)**.** Under the condition of $\|\nabla f_t(\mathbf{x})\| \le G$ for any $\mathbf{x} \in \mathcal{X}$ and $t \in [T]$, all online functions are L-smooth: $\|\nabla f_t(\mathbf{x}) - \nabla f_t(\mathbf{y})\| \le L\|\mathbf{x} - \mathbf{y}\|$ over $\mathcal{X}_+$, where $\mathcal{X}_+ \triangleq \{\mathbf{x} + \mathbf{z} \mid \mathbf{x} \in \mathcal{X}, \mathbf{z} \in G\mathbb{B}/L\}$ and $\mathbb{B} \triangleq \{\mathbf{x} \mid \|\mathbf{x}\| \le 1\}$ is a unit ball.

We note that Assumption 5 is reasonable, as many commonly used functions are globally smooth, such as the squared loss $f(\mathbf{x}) = \frac{1}{2}\|\mathbf{x}\|^2$ and the logistic loss $f(\mathbf{x}) = \log(1 + \exp(-\boldsymbol{\theta}^\top \mathbf{x}))$ for some $\boldsymbol{\theta} \in \mathbb{R}^d$. While the feasible domains of these functions are constrained to $\mathcal{X}$ (satisfying Assumption 4), they remain smooth over larger sets that extend beyond $\mathcal{X}$, thereby satisfying Assumption 5.

Building on the fact that Assumption 5 is a sufficient condition for Eq. (4.2) on $\mathcal{X}$, it can be directly obtained that Lemma 6 also holds under the relaxed Assumption 5.

4.2 Overall Algorithm and Regret Guarantee

In this section, we present the overall algorithm and its regret guarantees. Our algorithm adopts a two-layer online ensemble structure. Leveraging the new decomposition in Lemma 6, the meta algorithm does *not* require explicit control of its algorithmic stability; it only needs an optimistic second-order regret guarantee. Consequently, we employ Optimistic-Adapt-ML-Prod (Wei et al., 2016) as the meta algorithm. The base learners remain the same as those introduced in Section 2.3.

Meta Algorithm. UniGrad.Bregman employs Optimistic-Adapt-ML-Prod (Wei et al., 2016) as the meta algorithm to dynamically combine the base learners, in contrast to the complex MoM used in UniGrad.Correct. Optimistic-Adapt-ML-Prod is a simple algorithm that enjoys a second-order optimistic regret bound and admits closed-form weight updates. Specifically, the weight vector $\boldsymbol{p}_{t+1} \in \Delta_N$ is updated as follows: $\forall i \in [N]$,

$$\forall t \ge 1, W_{t,i} = \left(W_{t-1,i} \cdot \exp\left(\varepsilon_{t-1,i} r_{t,i} - \varepsilon_{t-1,i}^2 (r_{t,i} - m_{t,i})\right)\right)^{\frac{\varepsilon_{t,i}}{\varepsilon_{t-1,i}}}, \text{ and } W_{0,i} = \frac{1}{N}, \qquad (4.5)$$
$$p_{t+1,i} \propto \varepsilon_{t,i} \cdot \exp(\varepsilon_{t,i} m_{t+1,i}) \cdot W_{t,i},$$

where $W_{t,i}$ and $\varepsilon_{t,i}$ denote the potential variable and learning rate for the i-th base learner, respectively. The feedback loss vector $\boldsymbol{\ell}_t \in \mathbb{R}^N$ is configured as $\ell_{t,i} \triangleq \frac{1}{2GD}\langle \nabla f_t(\mathbf{x}_t), \mathbf{x}_{t,i}\rangle + \frac{1}{2} \in$

Algorithm 3 UniGrad.Bregman: Universal GV Regret by Extracted Bregman Divergence

Input: Base learner configurations $\{\mathcal{B}_i\}_{i\in[N]} \triangleq \{\mathcal{B}_i^{\text{sc}}\}_{i\in[N_{\text{sc}}]} \cup \{\mathcal{B}_i^{\exp}\}_{i\in[N_{\exp}]} \cup \mathcal{B}^{\text{c}}$

1: **Initialize:** $\mathcal{M}$ — meta learner Optimistic-Adapt-ML-Prod with $W_{0,i} = \frac{1}{N}$ for all $i \in [N]$
$\{\mathcal{B}_i\}_{i\in[N]}$ — base learners as specified in Section 2.3
2: **for** $t = 1$ **to** T **do**
3: Submit $\mathbf{x}_t = \sum_{i=1}^N p_{t,i}\mathbf{x}_{t,i}$, suffer $f_t(\mathbf{x}_t)$, and observe $\nabla f_t(\cdot)$
4: $\{\mathcal{B}_i\}_{i=1}^N$ update their own decisions to $\{\mathbf{x}_{t+1,i}\}_{i=1}^N$ using $\nabla f_t(\cdot)$
5: Calculate $\boldsymbol{m}_{t+1}$ (4.6) and $\boldsymbol{r}_t$ using $\{\mathbf{x}_{t,i}\}_{i=1}^N$, $\mathbf{x}_t$, $\nabla f_t(\mathbf{x}_t)$, and $\{\mathbf{x}_{t+1,i}\}_{i=1}^N$, send them to $\mathcal{M}$, and obtain $\boldsymbol{p}_{t+1} \in \Delta_N$ via Eq. (4.5) and Eq. (4.7)
6: **end for**

$[0,1]$, where $r_{t,i} = \langle \boldsymbol{p}_t, \boldsymbol{\ell}_t\rangle - \ell_{t,i}$ measures the instantaneous regret. The optimistic vector $\boldsymbol{m}_t \in \mathbb{R}^N$ is designed as[5]

$$m_{t,i} = \tfrac{1}{2GD}\langle \nabla f_{t-1}(\mathbf{x}_{t-1}), \mathbf{x}_t - \mathbf{x}_{t,i}\rangle \text{ for } i \in [N_{\text{c}}], \text{ and } m_{t,i} = 0 \text{ for } i \in [N_{\exp}] \cup [N_{\text{sc}}]. \quad (4.6)$$

The learning rate $\varepsilon_{t,i}$ is set as

$$\varepsilon_{t,i} = \min\left\{\frac{1}{8}, \sqrt{\frac{\log N}{\sum_{s=1}^t (r_{s,i} - m_{s,i})^2}}\right\}. \quad (4.7)$$

Algorithm 3 describes the overall update procedures of UniGrad.Bregman.

We now present the regret guarantees of UniGrad.Bregman, demonstrating that the algorithm achieves optimal gradient-variation regret without requiring prior knowledge of curvature information. The proof is provided in Appendix B.3.

Theorem 2. *Under Assumptions 1, 2, 5, by setting the learning rate of meta algorithm as Eq. (4.7),* UniGrad.Bregman *(Algorithm 3) achieves the following universal gradient-variation regret guarantees:*

$$\sum_{t=1}^T f_t(\mathbf{x}_t) - \min_{\mathbf{x}\in\mathcal{X}} \sum_{t=1}^T f_t(\mathbf{x}) \le \begin{cases} \mathcal{O}\left(\frac{1}{\lambda}\log V_T\right), & \textit{when } \{f_t\}_{t=1}^T \textit{ are } \lambda\textit{-strongly convex,} \\ \mathcal{O}\left(\frac{d}{\alpha}\log V_T\right), & \textit{when } \{f_t\}_{t=1}^T \textit{ are } \alpha\textit{-exp-concave,} \\ \mathcal{O}(\sqrt{V_T}), & \textit{when } \{f_t\}_{t=1}^T \textit{ are convex.} \end{cases}$$

Notably, UniGrad.Bregman improves the convex regret bound from $\mathcal{O}(\sqrt{V_T \log V_T})$ to the optimal $\mathcal{O}(\sqrt{V_T})$ (Yang et al., 2014), at the cost of requiring smoothness over a slightly larger domain. Even when the curvature α (or λ) is smaller than $1/T$, our algorithm still guarantees an $\mathcal{O}(\sqrt{V_T})$ bound, since exp-concave and strongly convex functions are also convex, and therefore our convex bound remains valid.

Finally, we compare UniGrad.Correct and UniGrad.Bregman in terms of their implications and applications. UniGrad.Correct is applicable to multi-player game settings (Syrgkanis

5. Though $\mathbf{x}_t$ is unknown when using $m_{t,i}$, we only need the scalar value of $\langle \nabla f_{t-1}(\mathbf{x}_{t-1}), \mathbf{x}_t\rangle$, which is bounded and can be efficiently solved via a one-dimensional fixed-point problem: $\langle \nabla f_{t-1}(\mathbf{x}_{t-1}), \mathbf{x}_t(z)\rangle = z$. $\mathbf{x}_t$ is a function of z because $\mathbf{x}_t$ relies on $p_{t,i}$, $p_{t,i}$ relies on $m_{t,i}$ and $m_{t,i}$ relies on z. Interested readers can refer to Section 3.3 of Wei et al. (2016) for more details.

et al., 2015; Zhang et al., 2022b) due to its stability property, which has been shown to be essential for achieving faster convergence in games (Syrgkanis et al., 2015). UniGrad.Bregman, by contrast, can be extended to the anytime setting, thanks to the simplicity and flexibility of its meta algorithm. Details of these extensions and applications are provided in Section 6. A more detailed comparison between UniGrad.Correct and UniGrad.Bregman will be discussed in Section 7.1.

5. One Gradient Query per Round

Though achieving favorable regret guarantees in Section 3 and Section 4, one caveat is that both UniGrad.Correct and UniGrad.Bregman require $\mathcal{O}(\log T)$ gradient queries per round because they need $\nabla f_t(\mathbf{x}_{t,i})$ for all $i \in [N]$, making them computationally inefficient when the gradient evaluation is costly, e.g., in nuclear norm optimization (Ji and Ye, 2009) and mini-batch optimization (Li et al., 2014). The same concern also appears in the approach of Zhang et al. (2022a), who provided small-loss and worst-case regret guarantees for universal online learning. By contrast, traditional algorithms such as OGD typically work under the *one-gradient* feedback setup, namely, they only require one gradient $\nabla f_t(\mathbf{x}_t)$ for the update. In light of this, it is natural to ask *whether we can design a universal algorithm that can maintain the desired regret guarantees while using only one gradient query per round.*

In this section, we provide an affirmative answer for this question via a dedicated surrogate optimization technique that implements base algorithms on carefully designed surrogate functions. We first present a general framework for the one-gradient algorithm in Section 5.1, and then instantiate it for UniGrad.Correct and UniGrad.Bregman algorithms, in Section 5.2 and Section 5.3, respectively.

5.1 A General Idea of Surrogate Optimization

In the following, we take λ-strongly convex functions as an example. To address this challenge, inspired by Wang et al. (2018), we propose an effective regret decomposition as follows. Specifically, let $\mathbf{x}^\star \in \arg\min_{\mathbf{x}\in\mathcal{X}} \sum_{t\in[T]} f_t(\mathbf{x})$ denote the optimal solution and $i^\star$ be the index of the best base learner with $\lambda_{i^\star} \le \lambda \le 2\lambda_{i^\star}$. We have

$$
\begin{aligned}
\textsc{Reg}_T &\le \sum_{t=1}^T \langle \nabla f_t(\mathbf{x}_t), \mathbf{x}_t - \mathbf{x}^\star\rangle - \frac{\lambda}{4}\sum_{t=1}^T \|\mathbf{x}_t - \mathbf{x}^\star\|^2 - \frac{1}{2}\sum_{t=1}^T \mathcal{D}_{f_t}(\mathbf{x}^\star, \mathbf{x}_t)\\
&\le \sum_{t=1}^T \langle \nabla f_t(\mathbf{x}_t), \mathbf{x}_t - \mathbf{x}^\star\rangle - \frac{\lambda_{i^\star}}{4}\sum_{t=1}^T \|\mathbf{x}_t - \mathbf{x}^\star\|^2 - \frac{1}{2}\sum_{t=1}^T \mathcal{D}_{f_t}(\mathbf{x}^\star, \mathbf{x}_t)\\
&\le \left[\sum_{t=1}^T \langle \nabla f_t(\mathbf{x}_t), \mathbf{x}_t - \mathbf{x}_{t,i^\star}\rangle - \frac{\lambda_{i^\star}}{4}\sum_{t=1}^T \|\mathbf{x}_t - \mathbf{x}_{t,i^\star}\|^2\right] - \frac{1}{2}\sum_{t=1}^T \mathcal{D}_{f_t}(\mathbf{x}^\star, \mathbf{x}_t)\\
&\quad + \left[\sum_{t=1}^T \left(\langle \nabla f_t(\mathbf{x}_t), \mathbf{x}_{t,i^\star}\rangle + \frac{\lambda_{i^\star}}{4}\|\mathbf{x}_{t,i^\star} - \mathbf{x}_t\|^2\right) - \sum_{t=1}^T \left(\langle \nabla f_t(\mathbf{x}_t), \mathbf{x}^\star\rangle + \frac{\lambda_{i^\star}}{4}\|\mathbf{x}^\star - \mathbf{x}_t\|^2\right)\right]\\
&= \underbrace{\left[\sum_{t=1}^T r_{t,i^\star} - \frac{\lambda_{i^\star}}{4}\sum_{t=1}^T \|\mathbf{x}_t - \mathbf{x}_{t,i^\star}\|^2\right]}_{\textsc{Meta-Reg}} + \underbrace{\left[\sum_{t=1}^T h^{\text{sc}}_{t,i^\star}(\mathbf{x}_{t,i^\star}) - \sum_{t=1}^T h^{\text{sc}}_{t,i^\star}(\mathbf{x}^\star)\right]}_{\textsc{Base-Reg}} - \frac{1}{2}\sum_{t=1}^T \mathcal{D}_{f_t}(\mathbf{x}^\star, \mathbf{x}_t).
\end{aligned}
$$

The first step follows from $\mathcal{D}_{f_t}(\mathbf{x}^\star, \mathbf{x}_t) \geq \frac{\lambda}{2}\|\mathbf{x}_t - \mathbf{x}^\star\|^2$ since $f_t(\cdot)$ is λ-strongly convex, and it preserves the Bregman divergence linearization-induced negative term for UniGrad.Bregman. The second step uses the definition of the best base learner (indexed by $i^\star$): $\lambda_{i^\star} \leq \lambda \leq 2\lambda_{i^\star}$. The third step inserts an intermediate term $\frac{\lambda_{i^\star}}{4}\sum_{t=1}^T \|\mathbf{x}_t - \mathbf{x}_{t,i^\star}\|^2$ and reorganizes the equation. The last step rewrites the equation by defining the following surrogate loss:

$$h_{t,i}^{\mathrm{sc}}(\mathbf{x}) \triangleq \langle \nabla f_t(\mathbf{x}_t), \mathbf{x}\rangle + \frac{\lambda_i}{4}\|\mathbf{x} - \mathbf{x}_t\|^2. \tag{5.1}$$

Note that the meta regret here is nearly identical to that in the multi-gradient setup, which can be optimized via algorithms with second-order regret guarantees, as demonstrated in Section 3 and Section 4, making it as flexible as Zhang et al. (2022a). Furthermore, the surrogate loss function in Eq. (5.1) requires *only one* gradient $\nabla f_t(\mathbf{x}_t)$, making it as efficient as van Erven and Koolen (2016).

Similarly, for α-exp-concave and convex functions, we define the surrogates

$$h_{t,i}^{\mathrm{exp}}(\mathbf{x}) \triangleq \langle \nabla f_t(\mathbf{x}_t), \mathbf{x}\rangle + \frac{\alpha_i}{4}\langle \nabla f_t(\mathbf{x}_t), \mathbf{x} - \mathbf{x}_t\rangle^2, \quad h_t^{\mathrm{c}}(\mathbf{x}) \triangleq \langle \nabla f_t(\mathbf{x}_t), \mathbf{x}\rangle. \tag{5.2}$$

In Section 5.2 and Section 5.3, we propose one-gradient improvements of UniGrad.Correct and UniGrad.Bregman, respectively, and demonstrate that additional novel analyses are required to achieve gradient-variation guarantees for the base regret, *defined on surrogates*.

As a byproduct, we show that this regret decomposition approach can be used to recover the minimax optimal worst-case universal guarantees using one gradient with a simple approach and analysis, with proof provided in Appendix C.2.

Proposition 2. *Under Assumptions 1 and 2, using the surrogate loss functions as defined in Eq. (5.1) and Eq. (5.2), and running Adapt-ML-Prod as the meta algorithm (by choosing $\boldsymbol{m}_t = \mathbf{0}$ in Eq. (4.5)) guarantees $\mathcal{O}(\frac{1}{\lambda}\log T)$, $\mathcal{O}(\frac{d}{\alpha}\log T)$ and $\mathcal{O}(\sqrt{T})$ regret bounds for strongly convex, exp-concave and convex functions, using one gradient per round.*

Remark 3. This result demonstrates that our surrogate optimization framework not only enables one-gradient universal algorithms but also provides a unified approach to recover classical minimax optimal bounds. The simplicity of the analysis compared to existing approaches highlights the power of the surrogate loss technique. ◁

5.2 UniGrad++.Correct: One-Gradient Improvement of UniGrad.Correct

To begin with, we recall the base regret definition with surrogates (we still take λ-strongly convex functions as an example):

$$\textsc{Base-Reg} = \left[\sum_{t=1}^T h_{t,i^\star}^{\mathrm{sc}}(\mathbf{x}_{t,i^\star}) - \sum_{t=1}^T h_{t,i^\star}^{\mathrm{sc}}(\mathbf{x}^\star)\right], \text{ where } h_{t,i}^{\mathrm{sc}}(\mathbf{x}) \triangleq \langle \nabla f_t(\mathbf{x}_t), \mathbf{x}\rangle + \frac{\lambda_i}{4}\|\mathbf{x} - \mathbf{x}_t\|^2.$$

For strongly convex gradient-variation regret minimization, the best known algorithm runs an initialization of the OOMD:

$$\mathbf{x}_{t,i} = \Pi_{\mathcal{X}}\left[\widehat{\mathbf{x}}_{t,i} - \eta_{t,i} M_{t,i}\right], \qquad \widehat{\mathbf{x}}_{t+1,i} = \Pi_{\mathcal{X}}\left[\widehat{\mathbf{x}}_{t,i} - \eta_{t,i}\nabla h_{t,i}^{\mathrm{sc}}(\mathbf{x}_{t,i})\right],$$

Algorithm 4 UniGrad++.Correct: One-Gradient Improvement of UniGrad.Correct

Input: Base learner configurations $\{\mathcal{B}_i\}_{i\in[N]} \triangleq \{\mathcal{B}_i^{\text{sc}}\}_{i\in[N_{\text{sc}}]} \cup \{\mathcal{B}_i^{\text{exp}}\}_{i\in[N_{\text{exp}}]} \cup \mathcal{B}^{\text{c}}$, algorithm parameters γ^{MID}, γ^{TOP}, and C_0

1: **Initialize:** $\mathcal{M}$ — meta learner MoM as shown in Algorithm 1
$\{\mathcal{B}_i\}_{i\in[N]}$ — base learners as specified in Section 2.3
$\{h_{t,i}^{\text{sc}}(\cdot)\}_{i\in[N]}$ — strongly convex surrogate losses as defined in Eq. (5.1)
2: **for** $t = 1$ **to** T **do**
3: Submit $\mathbf{x}_t = \sum_{i=1}^N p_{t,i}\mathbf{x}_{t,i}$, suffer $f_t(\mathbf{x}_t)$, and observe $\nabla f_t(\cdot)$
4: $\{\mathcal{B}_i^{\text{sc}}\}_{i\in[N_{\text{sc}}]}$, $\{\mathcal{B}_i^{\text{exp}}\}_{i\in[N_{\text{exp}}]}$, and $\mathcal{B}^{\text{c}}$ update their own decisions to $\{\mathbf{x}_{t+1,i}\}_{i=1}^N$ using surrogate losses of $\{h_{t,i}^{\text{sc}}(\cdot)\}_{\lambda_i\in\mathcal{H}^{\text{sc}}}$ (5.1), $\{h_{t,i}^{\text{exp}}(\cdot)\}_{\alpha_i\in\mathcal{H}^{\text{exp}}}$ (5.2), and $h_t^{\text{c}}(\cdot)$ (5.2)
5: Compute $\{\boldsymbol{\ell}_{t,j}^{\text{MID}}, \boldsymbol{m}_{t+1,j}^{\text{MID}}\}_{j=1}^M$ via Eq. (3.17), send to $\mathcal{M}$, get $\{\boldsymbol{q}_{t+1,j}^{\text{MID}}\}_{j=1}^M \in (\Delta_N)^M$
6: Compute $\boldsymbol{\ell}_t^{\text{TOP}}, \boldsymbol{m}_{t+1}^{\text{TOP}}$ via Eq. (3.16), send to $\mathcal{M}$, and obtain $\boldsymbol{q}_{t+1}^{\text{TOP}} \in \Delta_M$
7: Aggregate the final meta weights $\boldsymbol{p}_{t+1} \in \Delta_N$ via Eq. (3.12)
8: **end for**

where $\eta_{t,i}$ represents the step size. With appropriately chosen step sizes, the base learner achieves an optimistic bound of $\mathcal{O}(\log(\sum_{t\le T}\|\nabla h_{t,i}^{\text{sc}}(\mathbf{x}_{t,i}) - M_{t,i}\|^2))$ (e.g., Theorem 15 of Chiang et al. (2012)). Therefore, choosing the optimism as $M_{t,i} = \nabla h_{t-1,i}^{\text{sc}}(\mathbf{x}_{t-1,i})$ leads to an empirical gradient-variation bound $\mathcal{O}(\frac{1}{\lambda}\log \bar{V}_{T,i}^{\text{sc}})$ *defined on surrogates*, where $\bar{V}_{T,i}^{\text{sc}} \triangleq \sum_{t=2}^T \|\nabla h_{t,i}^{\text{sc}}(\mathbf{x}_t) - \nabla h_{t-1,i}^{\text{sc}}(\mathbf{x}_{t-1})\|^2$. To handle this term, we decompose it as

$$\begin{aligned}\bar{V}_{T,i}^{\text{sc}} &= \sum_{t=2}^T \left\|\nabla f_t(\mathbf{x}_t) + \frac{\lambda_i}{2}(\mathbf{x}_{t,i} - \mathbf{x}_t) - \nabla f_{t-1}(\mathbf{x}_{t-1}) - \frac{\lambda_i}{2}(\mathbf{x}_{t-1,i} - \mathbf{x}_{t-1})\right\|^2 \\ &\lesssim \sum_{t=2}^T \|\nabla f_t(\mathbf{x}_t) - \nabla f_{t-1}(\mathbf{x}_{t-1})\|^2 + \lambda_i^2 \sum_{t=2}^T \|\mathbf{x}_t - \mathbf{x}_{t-1}\|^2 + \lambda_i^2 \sum_{t=2}^T \|\mathbf{x}_{t,i} - \mathbf{x}_{t-1,i}\|^2,\end{aligned}$$

where $\nabla h_{t,i}^{\text{sc}}(\mathbf{x}) = \nabla f_t(\mathbf{x}_t) + \frac{\lambda_i}{2}(\mathbf{x} - \mathbf{x}_t)$ because of the definition of the strongly convex surrogate function (5.1). Notice that the above decomposition not only contains the desired gradient variation, but also includes the positive stability terms of base decisions $\|\mathbf{x}_{t,i} - \mathbf{x}_{t-1,i}\|^2$ and final decisions $\|\mathbf{x}_t - \mathbf{x}_{t-1}\|^2$. Fortunately, same as UniGrad.Correct in Section 3, these stability terms can be effectively addressed through our cancellation mechanism within the online ensemble, by adjusting the correction coefficients accordingly.

This efficient version is concluded in Algorithm 4, where the only algorithmic modification from Algorithm 2 is that in Line 4, base learners update on the carefully designed surrogate functions, not the original ones. A more detailed description of base learners' update rules are deferred to Appendix C.1 for self-containedness. We provide the regret guarantee below, which achieves the same guarantees as Theorem 1 with only one gradient per round. The proof is in Appendix C.3.

Theorem 3. *Under Assumptions 1, 2, 4, by setting*

$$C_0 = \max\left\{1, 8D, 4\gamma^{\text{TOP}}, 4D^2C_{11}, 16D^2C_{10}, 80D^3L^2 + 4D^2C_1\right\}, \tag{5.3}$$

$$\gamma^{\text{TOP}} = \max\left\{2D^2C_{11}, 8D^2C_{10}, 40D^3L^2 + 2D^2C_1\right\}, \gamma^{\text{MID}} = \max\left\{C_{11}, 4C_{10}, 20DL^2 + C_1\right\},$$

where $C_1 = 128(D^2L^2 + G^2)$*,* $C_{10} = 4L^2 + 32D^2G^2L^2 + 8G^4$*, and* $C_{11} = 128G^2(1 + L^2)$*,* UniGrad++.Correct *(Algorithm 4) achieves the following universal gradient-variation regret guarantees using only one gradient per round:*

$$\sum_{t=1}^{T} f_t(\mathbf{x}_t) - \min_{\mathbf{x}\in\mathcal{X}} \sum_{t=1}^{T} f_t(\mathbf{x}) \leq \begin{cases} \mathcal{O}\left(\frac{1}{\lambda}\log V_T\right), & \text{when } \{f_t\}_{t=1}^T \text{ are } \lambda\text{-strongly convex,} \\ \mathcal{O}\left(\frac{d}{\alpha}\log V_T\right), & \text{when } \{f_t\}_{t=1}^T \text{ are } \alpha\text{-exp-concave,} \\ \mathcal{O}(\sqrt{V_T \log V_T}), & \text{when } \{f_t\}_{t=1}^T \text{ are convex.} \end{cases}$$

5.3 UniGrad++.Bregman: One-Gradient Improvement of UniGrad.Bregman

In this part, we leverage the same idea of surrogate losses to improve the gradient query efficiency of UniGrad.Bregman. However, this becomes more challenging than that in Section 5.2, because the meta-base regret decomposition of Eq. (2.6) (we restate it below)

$$\text{Reg}_T = \left[\sum_{t=1}^{T} f_t(\mathbf{x}_t) - \sum_{t=1}^{T} f_t(\mathbf{x}_{t,i^\star})\right] + \left[\sum_{t=1}^{T} f_t(\mathbf{x}_{t,i^\star}) - \sum_{t=1}^{T} f_t(\mathbf{x}^\star)\right],$$

is not suitable anymore because this decomposition would require each base learner to access its own gradient $\nabla f_t(\mathbf{x}_{t,i})$ per round, which is not allowed in the one-gradient setup. Therefore, the decomposition in Lemma 6 becomes invalid, since the negative Bregman divergence term $\mathcal{D}_{f_t}(\mathbf{x}_{t,i^\star}, \mathbf{x}_t)$ in Eq. (4.4) becomes vacuous.

Empirical Gradient Variation Decomposition. To address this issue, we need to find a new regret decomposition that allows us to use only one gradient $\nabla f_t(\mathbf{x}_t)$ per round. Recall that the negative Bregman divergence term from linearization is algorithm-independent. Building on this observation, we propose a new decomposition for the empirical gradient variation by inserting algorithm-independent intermediate terms such as $\nabla f_t(\mathbf{x}^\star)$ and $\nabla f_{t-1}(\mathbf{x}^\star)$, where $\mathbf{x}^\star \in \arg\min_{\mathbf{x}\in\mathcal{X}} \sum_{t=1}^T f_t(\mathbf{x})$ and a more detailed derivation with constants is provided in Eq. (C.14).

$$\begin{aligned} \bar{V}_T &\lesssim \sum_{t=2}^{T} \left(\|\nabla f_t(\mathbf{x}_t) - \nabla f_t(\mathbf{x}^\star)\|^2 + \|\nabla f_t(\mathbf{x}^\star) - \nabla f_{t-1}(\mathbf{x}^\star)\|^2 + \|\nabla f_{t-1}(\mathbf{x}^\star) - \nabla f_{t-1}(\mathbf{x}_{t-1})\|^2\right) \\ &\overset{(4.2)}{\lesssim} L\sum_{t=2}^{T} \mathcal{D}_{f_t}(\mathbf{x}^\star, \mathbf{x}_t) + V_T + L\sum_{t=2}^{T} \mathcal{D}_{f_{t-1}}(\mathbf{x}^\star, \mathbf{x}_{t-1}) \leq 2L\sum_{t=1}^{T} \mathcal{D}_{f_t}(\mathbf{x}^\star, \mathbf{x}_t) + V_T, \end{aligned} \tag{5.4}$$

Consequently, to cancel the additional $\mathcal{D}_{f_t}(\mathbf{x}^\star, \mathbf{x}_t)$ while using only one gradient $\nabla f_t(\mathbf{x}_t)$ per round, we use the overall regret linearization below:

$$\sum_{t=1}^{T} f_t(\mathbf{x}_t) - \sum_{t=1}^{T} f_t(\mathbf{x}^\star) = \sum_{t=1}^{T} \langle \nabla f_t(\mathbf{x}_t), \mathbf{x}_t - \mathbf{x}^\star\rangle - \sum_{t=1}^{T} \mathcal{D}_{f_t}(\mathbf{x}^\star, \mathbf{x}_t). \tag{5.5}$$

Surrogate Empirical Gradient Variation. Furthermore, we show that additional novel analysis is required to handle the empirical gradient variation *defined on surrogates*. Again,

Algorithm 5 UniGrad++.Bregman: One-Gradient Improvement of UniGrad.Bregman

Input: Base learner configurations $\{\mathcal{B}_i\}_{i\in[N]} \triangleq \{\mathcal{B}_i^{\text{sc}}\}_{i\in[N_{\text{sc}}]} \cup \{\mathcal{B}_i^{\exp}\}_{i\in[N_{\exp}]} \cup \mathcal{B}^{\text{c}}$

1: **Initialize**: $\mathcal{M}$ — meta learner Optimistic-Adapt-ML-Prod with $W_{0,i} = \frac{1}{N}$ for all $i \in [N]$
$\{\mathcal{B}_i\}_{i\in[N]}$ — base learners as specified in Section 2.3

2: **for** $t = 1$ **to** T **do**

3: Submit $\mathbf{x}_t = \sum_{i=1}^N p_{t,i}\mathbf{x}_{t,i}$, suffer $f_t(\mathbf{x}_t)$, and observe $\nabla f_t(\cdot)$

4: $\{\mathcal{B}_i^{\text{sc}}\}_{i\in[N_{\text{sc}}]}$, $\{\mathcal{B}_i^{\exp}\}_{i\in[N_{\exp}]}$, $\mathcal{B}^{\text{c}}$ update their own decisions to $\{\mathbf{x}_{t+1,i}\}_{i=1}^N$ using surrogate losses of $\{h_{t,i}^{\text{sc}}(\cdot)\}_{\lambda_i\in\mathcal{H}^{\text{sc}}}$ (5.1), $\{h_{t,i}^{\exp}(\cdot)\}_{\alpha_i\in\mathcal{H}^{\exp}}$ (5.2), and $h_t^{\text{c}}(\cdot)$ (5.2)

5: Calculate $\boldsymbol{m}_{t+1}$ (4.6) and $\boldsymbol{r}_t$ using $\{\mathbf{x}_{t,i}\}_{i=1}^N$, $\mathbf{x}_t$, $\nabla f_t(\mathbf{x}_t)$, and $\{\mathbf{x}_{t+1,i}\}_{i=1}^N$, send them to $\mathcal{M}$, and obtain $\boldsymbol{p}_{t+1} \in \Delta_N$

6: **end for**

we take λ-strongly convex functions as an example and provide the following decomposition of the empirical gradient on surrogates:

$$\begin{aligned}
D_{T,i}^{\text{sc}} &= \sum_{t=2}^T \|\nabla h_{t,i}^{\text{sc}}(\mathbf{x}_{t,i}) - \nabla h_{t-1,i}^{\text{sc}}(\mathbf{x}_{t-1,i})\|^2 \\
&= \sum_{t=2}^T \left\|\nabla f_t(\mathbf{x}_t) - \nabla f_{t-1}(\mathbf{x}_{t-1}) + \frac{\lambda_i}{2}(\mathbf{x}_{t,i} - \mathbf{x}_t) - \frac{\lambda_i}{2}(\mathbf{x}_{t-1,i} - \mathbf{x}_{t-1})\right\|^2 \\
&\lesssim \sum_{t=2}^T \|\nabla f_t(\mathbf{x}_t) - \nabla f_{t-1}(\mathbf{x}_{t-1})\|^2 + \sum_{t=2}^T \|\mathbf{x}_{t,i} - \mathbf{x}_t\|^2 + \sum_{t=2}^T \|\mathbf{x}_{t-1,i} - \mathbf{x}_{t-1}\|^2 \\
&\lesssim \sum_{t=2}^T \|\nabla f_t(\mathbf{x}_t) - \nabla f_{t-1}(\mathbf{x}_{t-1})\|^2 + \sum_{t=2}^T \|\mathbf{x}_{t,i} - \mathbf{x}_t\|^2.
\end{aligned}$$

In the third step, instead of controlling $(\mathbf{x}_{t,i} - \mathbf{x}_t) - (\mathbf{x}_{t-1,i} - \mathbf{x}_{t-1})$ per round, which requires analyzing the stability term $\|\mathbf{x}_t - \mathbf{x}_{t-1}\|^2$ directly, we deal with the additional surrogate-induced terms by *aggregation over the time horizon*, using a similar idea in our unifying optimism design in Lemma 3. Consequently, this term can be canceled out by the curvature-induced negative term from the meta regret, as shown in Eq. (2.8). For this cancellation to occur, appropriate coefficients are needed, which are provided in the detailed proofs and are omitted here for clarity.[6]

The empirical gradient variation defined on the original functions, i.e., $\|\nabla f_t(\mathbf{x}_t) - \nabla f_{t-1}(\mathbf{x}_{t-1})\|^2$, can be decomposed as shown in Eq. (5.4) and canceled by the negative Bregman divergence terms from linearization as presented in Eq. (5.5).

This simple and novel analysis eliminates the need to control the overall algorithmic stability $\|\mathbf{x}_t - \mathbf{x}_{t-1}\|^2$ required by UniGrad.Correct, and is essential for achieving the optimal regret guarantees, as provided in the following, where the proof is deferred to Appendix C.4.

Theorem 4. *Under Assumptions 1, 2, 5, by setting the learning rate of meta algorithm as Eq. (4.7),* UniGrad++.Bregman *(Algorithm 5) achieves the following universal gradient-*

6. For strongly convex functions, a simpler choice would be $M_{t,i} = \nabla f_{t-1}(\mathbf{x}_{t-1})$ to allow simpler surrogate-induced terms. We choose the gradient of the last round as the optimism since this is the the only choice at present to achieve a gradient-variation regret for exp-concave functions (Chiang et al., 2012).

variation regret guarantees using only one gradient per round:

$$\sum_{t=1}^{T} f_t(\mathbf{x}_t) - \min_{\mathbf{x}\in\mathcal{X}} \sum_{t=1}^{T} f_t(\mathbf{x}) \leq \begin{cases} \mathcal{O}\left(\frac{1}{\lambda}\log V_T\right), & \textit{when } \{f_t\}_{t=1}^T \textit{ are } \lambda\textit{-strongly convex,} \\ \mathcal{O}\left(\frac{d}{\alpha}\log V_T\right), & \textit{when } \{f_t\}_{t=1}^T \textit{ are } \alpha\textit{-exp-concave,} \\ \mathcal{O}(\sqrt{V_T}), & \textit{when } \{f_t\}_{t=1}^T \textit{ are convex.} \end{cases}$$

6. Implications, Applications, and Extension

In this section, we demonstrate the effectiveness of our methods through their implications for small-loss and gradient-variance regret in Section 6.1, as well as their applications to the Stochastically Extended Adversarial (SEA) model (in Section 6.2) and online games (in Section 6.3). Finally, in Section 6.4, we establish optimal universal regret without requiring prior knowledge of the time horizon T through an anytime variant of our method.

6.1 Implications to Small-Loss and Gradient-Variance Bounds

In this subsection, we demonstrate that our universal gradient-variation regret bounds naturally yield other problem-dependent quantities such as small-loss (Srebro et al., 2010; Orabona et al., 2012) and gradient-variance (Hazan and Kale, 2008, 2009) regret bounds directly through the analysis *without* any algorithmic modifications. This demonstrates that our methods can capture the complexity of the online learning problem from multiple perspectives, providing a more comprehensive understanding of the problem's behavior.

Specifically, the small-loss and gradient-variance quantities are formally defined as:

$$\begin{aligned} F_T &\triangleq \min_{\mathbf{x}\in\mathcal{X}} \sum_{t=1}^{T} f_t(\mathbf{x}) - \sum_{t=1}^{T} \min_{\mathbf{x}\in\mathcal{X}_+} f_t(\mathbf{x}), \quad \mathcal{X}_+ \triangleq \left\{\mathbf{x}+\mathbf{z} \mid \mathbf{x}\in\mathcal{X}, \mathbf{z}\in \tfrac{G}{L}\cdot\mathbb{B}\right\} \\ W_T &\triangleq \sup_{\{\mathbf{x}_1,\ldots,\mathbf{x}_T\}\in\mathcal{X}} \left\{\sum_{t=1}^{T} \|\nabla f_t(\mathbf{x}_t) - \boldsymbol{\mu}_T\|^2\right\}, \quad \boldsymbol{\mu}_T \triangleq \frac{1}{T}\sum_{t=1}^{T} \nabla f_t(\mathbf{x}_t), \end{aligned} \tag{6.1}$$

where $\mathcal{X}_+$ is a superset of the original domain $\mathcal{X}$ defined in Assumption 5 and μ_T represents the average gradient.

In what follows, we demonstrate that both the small-loss and gradient-variance quantities can be derived from the empirical gradient variation through standard analytical techniques. Specifically, for the small-loss quantity, we utilize the self-bounding property $\|\nabla f(\mathbf{x})\|_2^2 \leq 4L(f(\mathbf{x}) - \min_{\mathbf{x}\in\mathcal{X}_+} f(\mathbf{x}))$ for any L-smooth function $f : \mathcal{X}_+ \to \mathbb{R}$ and any $\mathbf{x}\in\mathcal{X}_+$,[7] which yields

$$\begin{aligned} \bar{V}_T &\leq 2\sum_{t=2}^{T} \|\nabla f_t(\mathbf{x}_t)\|^2 + 2\sum_{t=2}^{T} \|\nabla f_{t-1}(\mathbf{x}_{t-1})\|^2 \leq 4\sum_{t=1}^{T} \|\nabla f_t(\mathbf{x}_t)\|^2 \\ &\leq 16L\left(\sum_{t=1}^{T} f_t(\mathbf{x}_t) - \min_{\mathbf{x}\in\mathcal{X}_+} f_t(\mathbf{x})\right). \end{aligned} \tag{6.2}$$

7. This more restricted self-bounding property can be derived using the arguments in Appendix B.2.

Note that the right-hand side of Eq. (6.2) can be directly transformed to the small-loss quantity using standard techniques (Srebro et al., 2010; Orabona et al., 2012).

Next, we demonstrate that the gradient-variance quantity can be derived from the empirical gradient variation through a standard analytical technique:

$$\begin{aligned}\bar{V}_T = \sum_{t=2}^{T} \|\nabla f_t(\mathbf{x}_t) - \nabla f_{t-1}(\mathbf{x}_{t-1})\|^2 &\le 2\sum_{t=2}^{T} \|\nabla f_t(\mathbf{x}_t) - \boldsymbol{\mu}_T\|^2 + 2\sum_{t=2}^{T} \|\nabla f_{t-1}(\mathbf{x}_{t-1}) - \boldsymbol{\mu}_T\|^2 \\ &\le 4\sum_{t=1}^{T} \|\nabla f_t(\mathbf{x}_t) - \boldsymbol{\mu}_T\|^2 \le 4W_T. \end{aligned} \tag{6.3}$$

To conclude, our universal gradient-variation regret can directly imply universal small-loss and gradient-variance guarantees without any algorithmic modifications. We present the corresponding guarantees for UniGrad++.Correct and UniGrad++.Bregman below. The proofs are deferred to Appendix D.1 and Appendix D.2.

Corollary 1. *Under Assumptions 1, 2, 5, by setting*

$$\begin{aligned} C_0 &= \max\left\{1, 8D, 4\gamma^{\text{TOP}}, 512D^2G^2, 128D^2G^4\right\}, \\ \gamma^{\text{TOP}} &= \max\left\{256D^2G^2, 64D^2G^4\right\}, \quad \gamma^{\text{MID}} = \max\left\{128G^2, 32G^4\right\}, \end{aligned} \tag{6.4}$$

UniGrad++.Correct *(Algorithm 4) achieves the following universal regret guarantees:*

$$\text{Reg}_T \le \begin{cases} \mathcal{O}\left(\min\left\{\frac{1}{\lambda}\log F_T,\ \frac{1}{\lambda}\log W_T\right\}\right), & \text{when } \{f_t\}_{t=1}^T \text{ are } \lambda\text{-strongly convex,} \\ \mathcal{O}\left(\min\left\{\frac{d}{\alpha}\log F_T,\ \frac{d}{\alpha}\log W_T\right\}\right), & \text{when } \{f_t\}_{t=1}^T \text{ are } \alpha\text{-exp-concave,} \\ \mathcal{O}\left(\min\left\{\sqrt{F_T\log F_T},\ \sqrt{W_T\log W_T}\right\}\right), & \text{when } \{f_t\}_{t=1}^T \text{ are convex.} \end{cases}$$

Corollary 2. *Under Assumptions 1, 2, 5, by setting the learning rate of meta algorithm as Eq. (4.7),* UniGrad++.Bregman *(Algorithm 5) achieves the following universal regret:*

$$\text{Reg}_T \le \begin{cases} \mathcal{O}\left(\min\left\{\frac{1}{\lambda}\log F_T,\ \frac{1}{\lambda}\log W_T\right\}\right), & \text{when } \{f_t\}_{t=1}^T \text{ are } \lambda\text{-strongly convex,} \\ \mathcal{O}\left(\min\left\{\frac{d}{\alpha}\log F_T,\ \frac{d}{\alpha}\log W_T\right\}\right), & \text{when } \{f_t\}_{t=1}^T \text{ are } \alpha\text{-exp-concave,} \\ \mathcal{O}\left(\min\left\{\sqrt{F_T},\ \sqrt{W_T}\right\}\right), & \text{when } \{f_t\}_{t=1}^T \text{ are convex.} \end{cases}$$

We would like to emphasize that the hyper-parameter configurations in Eq. (6.4) is employed only for the small-loss and gradient-variance regret. To achieve *best-of-three-worlds* guarantees in terms of $\min\{V_T, F_T, W_T\}$, the hyper-parameters of $C_0, \gamma^{\text{TOP}}, \gamma^{\text{MID}}$ should be set as the union of configurations in Eq. (5.3) and Eq. (6.4). Furthermore, because the hyper-parameter setups of UniGrad++.Bregman for this problem are the same for those for the gradient-variation regret (in Theorem 4), UniGrad++.Bregman directly enjoys *best-of-three-worlds* guarantees.

Table 2: Comparisons of our results with existing ones. The second column presents the regret bounds, where $\sigma_{1:T}^2$ and $\Sigma_{1:T}^2$ represent the stochastic and adversarial statistics of the SEA problem. The last column indicates whether the results can be achieved by a single algorithm (i.e., suitable in the universal setup). UniGrad++.Correct suffers and additional logarithmic factor compared with the best known guarantees of Chen et al. (2024), while UniGrad++.Bregman achieves exactly the same state-of-the-art guarantees using one single algorithm.

Method	**Regret Bounds**			**Single Algorithm?**
	Strongly Convex	Exp-concave	Convex	
Sachs et al. (2022)	$\mathcal{O}\left(\frac{1}{\lambda}(\sigma_{\max}^2+\Sigma_{\max}^2)\log T\right)$	N/A	$\mathcal{O}\left(\sqrt{\sigma_{1:T}^2+\Sigma_{1:T}^2}\right)$	✗
Chen et al. (2024)	$\mathcal{O}\left(\frac{1}{\lambda}(\sigma_{\max}^2+\Sigma_{\max}^2)\log\left(\frac{\sigma_{1:T}^2+\Sigma_{1:T}^2}{\sigma_{\max}^2+\Sigma_{\max}^2}\right)\right)$	$\mathcal{O}\left(\frac{d}{\alpha}\log(\sigma_{1:T}^2+\Sigma_{1:T}^2)\right)$	$\mathcal{O}\left(\sqrt{\sigma_{1:T}^2+\Sigma_{1:T}^2}\right)$	✗
Sachs et al. (2023)	$\mathcal{O}\left(\frac{1}{\lambda}(\sigma_{\max}^2+\Sigma_{\max}^2+D^2L^2)\log^2 T\right)$	N/A	$\mathcal{O}(\sqrt{T\log T})$	✓
UniGrad++.Correct	$\mathcal{O}\left(\frac{1}{\lambda}(\sigma_{\max}^2+\Sigma_{\max}^2)\log(\sigma_{1:T}^2+\Sigma_{1:T}^2)\right)$	$\mathcal{O}\left(\frac{d}{\alpha}\log(\sigma_{1:T}^2+\Sigma_{1:T}^2)\right)$	$\mathcal{O}\left(\sqrt{(\sigma_{1:T}^2+\Sigma_{1:T}^2)\log(\sigma_{1:T}^2+\Sigma_{1:T}^2)}\right)$	✓
UniGrad++.Bregman	$\mathcal{O}\left(\frac{1}{\lambda}(\sigma_{\max}^2+\Sigma_{\max}^2)\log\left(\frac{\sigma_{1:T}^2+\Sigma_{1:T}^2}{\sigma_{\max}^2+\Sigma_{\max}^2}\right)\right)$	$\mathcal{O}\left(\frac{d}{\alpha}\log(\sigma_{1:T}^2+\Sigma_{1:T}^2)\right)$	$\mathcal{O}\left(\sqrt{\sigma_{1:T}^2+\Sigma_{1:T}^2}\right)$	✓

6.2 Application to Stochastically Extended Adversarial (SEA) Model

Stochastically extended adversarial (SEA) model (Sachs et al., 2022) interpolates between stochastic and adversarial online convex optimization. Formally, it assumes that the online function $f_t(\cdot)$ is sampled stochastically from an adversarially chosen distribution $\mathfrak{D}_t$. Denoting by $F_t(\cdot) \triangleq \mathbb{E}_{f_t\sim\mathfrak{D}_t}[f_t(\cdot)]$ the expected function, two terms capture the essential characteristics of SEA model:

$$\sigma_{1:T}^2 \triangleq \sum_{t=1}^T \max_{\mathbf{x}\in\mathcal{X}} \mathbb{E}_{f_t\sim\mathfrak{D}_t}\left[\|\nabla f_t(\mathbf{x}) - \nabla F_t(\mathbf{x})\|^2\right], \quad \Sigma_{1:T}^2 \triangleq \mathbb{E}\left[\sum_{t=1}^T \sup_{\mathbf{x}\in\mathcal{X}} \|\nabla F_t(\mathbf{x}) - \nabla F_{t-1}(\mathbf{x})\|^2\right],$$

where $\sigma_{1:T}^2$ is the variance in sampling $f_t(\cdot)$ from $\mathfrak{D}_t(\cdot)$ and $\Sigma_{1:T}^2$ is the variation of $\{F_t(\cdot)\}_{t\in[T]}$. Accordingly, we define the per-round maximum versions of the above quantities as

$$\sigma_{\max}^2 \triangleq \max_{t\in[T]}\max_{\mathbf{x}\in\mathcal{X}} \mathbb{E}_{f_t\sim\mathfrak{D}_t}\left[\|\nabla f_t(\mathbf{x}) - \nabla F_t(\mathbf{x})\|^2\right], \quad \Sigma_{\max}^2 \triangleq \max_{t\in[T]}\sup_{\mathbf{x}\in\mathcal{X}} \|\nabla F_t(\mathbf{x}) - \nabla F_{t-1}(\mathbf{x})\|^2.$$

For the SEA problem, Sachs et al. (2022) pioneered the study of the SEA model. For smooth expected functions $\{F_t(\cdot)\}_{t=1}^T$, they established the optimal $\mathcal{O}(\sqrt{\sigma_{1:T}^2+\Sigma_{1:T}^2})$ regret for convex expected functions and $\mathcal{O}(\frac{1}{\lambda}(\sigma_{\max}^2+\Sigma_{\max}^2)\log T)$ in the strongly convex case. Subsequently, Chen et al. (2024) enhanced the strongly convex regret to $\mathcal{O}(\frac{1}{\lambda}(\sigma_{\max}^2+\Sigma_{\max}^2)\log((\sigma_{1:T}^2+\Sigma_{1:T}^2)/(\sigma_{\max}^2+\Sigma_{\max}^2)))$ and derived a new $\mathcal{O}(\frac{d}{\alpha}\log(\sigma_{1:T}^2+\Sigma_{1:T}^2))$ regret bound for exp-concave individual functions $\{f_t(\cdot)\}_{t=1}^T$.

The gradient variation is essential in connecting the stochastic and adversarial optimization in the SEA problem (Chen et al., 2024, Lemma 3). To see this, we can decompose the empirical gradient variation as:

$$\mathbb{E}\left[\sum_{t=2}^T \|\nabla f_t(\mathbf{x}_t) - \nabla f_{t-1}(\mathbf{x}_{t-1})\|^2\right] \le 4L^2\mathbb{E}\left[\sum_{t=2}^T \|\mathbf{x}_t - \mathbf{x}_{t-1}\|^2\right] + 8\sigma_{1:T}^2 + 4\Sigma_{1:T}^2 + \mathcal{O}(1), \quad (6.5)$$

which not only consists of the stochastic variation $\sigma_{1:T}^2$ and the adversarial variation $\Sigma_{1:T}^2$, but also the algorithmic stability $\|\mathbf{x}_t - \mathbf{x}_{t-1}\|^2$. And the last term can be perfectly handled by UniGrad++.Correct as introduced in Section 3 and Section 5.2.

For UniGrad++.Bregman to solve the SEA problem, since it cannot directly deal with the stability terms of $\|\mathbf{x}_t - \mathbf{x}_{t-1}\|^2$, we provide it a different decomposition of the empirical gradient variation to let negative Bregman divergence terms in the analysis of UniGrad++.Bregman to take effect. Specifically, we decompose it as

$$\mathbb{E}\left[\sum_{t=2}^{T} \|\nabla f_t(\mathbf{x}_t) - \nabla f_{t-1}(\mathbf{x}_{t-1})\|^2\right] \le 10\sigma_{1:T}^2 + 5\Sigma_{1:T}^2 + 20LE\left[\sum_{t=1}^{T} \mathcal{D}_{F_t}(\mathbf{x}^\star, \mathbf{x}_t)\right], \tag{6.6}$$

where the positive Bregman divergence terms can be canceled accordingly. A detailed derivation of this inequality is deferred to Eq. (D.25) in Appendix D.4.

Therefore, universal gradient-variation regret can be applied to in the SEA problem, therefore solving a major open problem from Chen et al. (2024) about whether it is possible to get rid of different parameter configurations and obtain universal guarantees. In the following, we show that our approaches of UniGrad++.Correct and UniGrad++.Bregman can be both directly applied and achieve almost the same guarantees as those in Chen et al. (2024), with a *single* algorithm. We conclude our results in Theorem 5 and Theorem 6 below and the proofs can be found in Appendix D.3 and Appendix D.4.

Theorem 5. *Under Assumptions 1, 2, 4, by setting*

$$C_0 = \max\left\{1, 8D, 4\gamma^{\text{TOP}}, 8D^2C_{24}, 8D^2C_{23}, 8D^2C_{25}\right\},$$
$$\gamma^{\text{TOP}} = \max\left\{4D^2C_{24}, 8D^2C_{23}, 4D^2\left(20DL^2 + 64G^2 + 128D^2L^2\right)\right\},$$
$$\gamma^{\text{MID}} = \max\left\{2C_{24}, 4C_{23}, 20DL^2 + 64G^2 + 128D^2L^2\right\},$$

where $C_{23} = 8L^2 + 64D^2G^2L^2 + 8G^4$, $C_{24} = 64D^2(1+L^2)^2$, *and* $C_{25} = 20DL^2 + \frac{64G^2}{Z} + 128D^2L^2$, UniGrad++.Correct *(Algorithm 4) achieves the following universal regret:*

$$\text{REG}_T \le \begin{cases} \mathcal{O}\left(\frac{1}{\lambda}(\sigma_{\max}^2 + \Sigma_{\max}^2)\log\left(\frac{\sigma_{1:T}^2+\Sigma_{1:T}^2}{\sigma_{\max}^2+\Sigma_{\max}^2}\right)\right), & \text{when } \{F_t\}_{t=1}^T \text{ are } \lambda\text{-strongly convex},\\ \mathcal{O}\left(\frac{d}{\alpha}\log(\sigma_{1:T}^2 + \Sigma_{1:T}^2)\right), & \text{when } \{f_t\}_{t=1}^T \text{ are } \alpha\text{-exp-concave},\\ \mathcal{O}\left(\sqrt{(\sigma_{1:T}^2 + \Sigma_{1:T}^2)\log(\sigma_{1:T}^2 + \Sigma_{1:T}^2)}\right), & \text{when } \{F_t\}_{t=1}^T \text{ are convex.}\end{cases}$$

Theorem 6. *Under Assumptions 1, 2, 5, by setting the learning rate of meta algorithm as Eq. (4.7),* UniGrad++.Bregman *(Algorithm 5) achieves the following universal gradient-variation regret guarantees:*

$$\text{REG}_T \le \begin{cases} \mathcal{O}\left(\frac{1}{\lambda}(\sigma_{\max}^2 + \Sigma_{\max}^2)\log\left(\frac{\sigma_{1:T}^2+\Sigma_{1:T}^2}{\sigma_{\max}^2+\Sigma_{\max}^2}\right)\right), & \text{when } \{F_t\}_{t=1}^T \text{ are } \lambda\text{-strongly convex},\\ \mathcal{O}\left(\frac{d}{\alpha}\log(\sigma_{1:T}^2 + \Sigma_{1:T}^2)\right), & \text{when } \{f_t\}_{t=1}^T \text{ are } \alpha\text{-exp-concave},\\ \mathcal{O}\left(\sqrt{\sigma_{1:T}^2 + \Sigma_{1:T}^2}\right), & \text{when } \{F_t\}_{t=1}^T \text{ are convex.}\end{cases}$$

Remark 4. Sachs et al. (2023) also considered the problem of universal learning and obtained an $\mathcal{O}(\sqrt{T\log T})$ regret for convex functions and an $\mathcal{O}((\sigma_{\max}^2 + \Sigma_{\max}^2 + D^2L^2)\log^2 T)$ regret for strongly convex functions simultaneously. We conclude the existing results in

Table 2. Our results are better than theirs in two aspects: *(i)* for strongly convex and convex functions, our guarantees are adaptive with the problem-dependent quantities $\sigma_{1:T}$ and $\Sigma_{1:T}$ while theirs depends on the time horizon T; and *(ii)* our algorithm achieves an additional guarantee for exp-concave functions. ◁

Remark 5. Theorem 5 and Theorem 6 require the exp-concavity of the individual function $f_t(\cdot)$ rather than the expected function $F_t(\cdot)$. This assumption is also used by Chen et al. (2024) and common in the studies of stochastic exp-concave optimization (Mahdavi et al., 2015; Koren and Levy, 2015). ◁

6.3 Application to Faster-Rate Convergence in Online Games

Multi-player online games (Cesa-Bianchi and Lugosi, 2006) is a versatile model that depicts the interaction of multiple players over time. Since each player is facing similar players like herself, the theoretical guarantees, e.g., the summation of all players' regret, can be better than the minimax optimal $\mathcal{O}(\sqrt{T})$ in adversarial environments, thus achieving *faster rates*.

The pioneering works of Rakhlin and Sridharan (2013b) and Syrgkanis et al. (2015) investigated optimistic algorithms in multi-player online games and illuminated the importance of the gradient variation. Specifically, Syrgkanis et al. (2015) showed that optimistic algorithms, such as OOMD or optimistic follow the regularized leader (Shalev-Shwartz and Singer, 2007; Shalev-Shwartz, 2007), possess a specific property known as "Regret bounded by Variation in Utilities" (RVU) property.

Definition 3 (RVU Property)**.** An algorithm with a parameter $\eta > 0$ satisfies the RVU property if there exist constants $\alpha, \beta, \gamma > 0$ such that the regret $\textsc{Reg}_T$ on decision sequence $\{\mathbf{x}_t\}_{t=1}^T$ and gradient sequence $\{\mathbf{g}_t\}_{t=1}^T$ is bounded by

$$\textsc{Reg}_T \le \frac{\alpha}{\eta} + \beta\eta \sum_{t=2}^{T} \|\mathbf{g}_t - \mathbf{g}_{t-1}\|_\infty^2 - \frac{\gamma}{\eta}\sum_{t=2}^{T} \|\mathbf{x}_t - \mathbf{x}_{t-1}\|_1^2. \tag{6.7}$$

To illustrate the usefulness of the RVU property, we consider a simple bilinear zero-sum game of $\mathbf{x}^\top A\mathbf{y}$ where $\mathbf{x}, \mathbf{y} \in \Delta_d$ and $\max_{i,j\in[d]} |A_{i,j}| \le 1$. In this case, the gradients of the $\mathbf{x}$-player are given by $\mathbf{g}_t^{\mathbf{x}} = A\mathbf{y}_t$ for $t \in [T]$, which implies $\sum_{t=2}^T \|\mathbf{g}_t^{\mathbf{x}} - \mathbf{g}_{t-1}^{\mathbf{x}}\|_\infty^2 = \sum_{t=2}^T \|A\mathbf{y}_t - A\mathbf{y}_{t-1}\|_\infty^2 \le \sum_{t=2}^T \|\mathbf{y}_t - \mathbf{y}_{t-1}\|_1^2$. Similarly, for the $\mathbf{y}$-player, we have $\sum_{t=2}^T \|\mathbf{g}_t^{\mathbf{y}} - \mathbf{g}_{t-1}^{\mathbf{y}}\|_\infty^2 \le \sum_{t=2}^T \|\mathbf{x}_t - \mathbf{x}_{t-1}\|_1^2$. By setting both learners' learning rates to $\eta^{\mathbf{x}} = \eta^{\mathbf{y}} = \sqrt{\gamma/\beta}$, the sum of the two players' regrets can be bounded by

$$\begin{aligned}\textsc{Reg}_T^{\mathbf{x}} + \textsc{Reg}_T^{\mathbf{y}} \le \frac{\alpha}{\eta^{\mathbf{x}}} + \frac{\alpha}{\eta^{\mathbf{y}}} + (\beta\eta^{\mathbf{x}} - \frac{\gamma}{\eta^{\mathbf{y}}}) \sum_{t=2}^{T} \|\mathbf{x}_t - \mathbf{x}_{t-1}\|_1^2 \\ + (\beta\eta^{\mathbf{y}} - \frac{\gamma}{\eta^{\mathbf{x}}}) \sum_{t=2}^{T} \|\mathbf{y}_t - \mathbf{y}_{t-1}\|_1^2 \le \mathcal{O}(1).\end{aligned}$$

This bound, in turn, enables the efficient computation of Nash equilibria.

The above results assume that the players are *honest*, i.e., they agree to run the same algorithm. In the *dishonest* case, where there exist players who do not follow the agreed protocol, the problem degenerates to two separate online adversarial convex optimization

Algorithm 6 UniGrad++.Correct for $\mathbf{x}$-player

Input: Base learner configurations $\{\mathcal{B}_i\}_{i\in[N]} \triangleq \{\mathcal{B}_i^{\text{sc}}\}_{i\in[N_{\text{sc}}]} \cup \mathcal{B}^{\text{c}}$, algorithm parameters γ^{MID}, γ^{TOP}, and C_0

Initialize: $\mathcal{M}$ — meta learner MoM as shown in Algorithm 1
$\{\mathcal{B}_i\}_{i\in[N]}$ — base learners as specified in Section 2.3

for $t = 1$ **to** T **do**

Submit $\mathbf{x}_t = \sum_{i=1}^N p_{t,i}\mathbf{x}_{t,i}$, suffer $f_t(\mathbf{x}, \mathbf{y})$, and observe $\mathbf{g}_t^{\mathbf{x}}$

$\{\mathcal{B}_i^{\text{sc}}\}_{i\in[N_{\text{sc}}]}$ and $\mathcal{B}^{\text{c}}$ update their own decisions to $\{\mathbf{x}_{t+1,i}\}_{i=1}^N$ using surrogate losses of $\{h_{t,i}^{\text{sc}}(\cdot)\}_{\lambda_i\in\mathcal{H}^{\text{sc}}}$ (5.1) and $h_t^{\text{c}}(\cdot)$ (5.2)

Compute $\{\boldsymbol{\ell}_{t,j}^{\text{MID}}, \boldsymbol{m}_{t+1,j}^{\text{MID}}\}_{j=1}^M$ via Eq. (3.17) with gradients of $\nabla f_t(\cdot) = \mathbf{g}_t^{\mathbf{x}}$, send to $\mathcal{M}$, get $\{\boldsymbol{q}_{t+1,j}^{\text{MID}}\}_{j=1}^M \in (\Delta_N)^M$

Compute $\boldsymbol{\ell}_t^{\text{TOP}}, \boldsymbol{m}_{t+1}^{\text{TOP}}$ via Eq. (3.16), send to $\mathcal{M}$, and obtain $\boldsymbol{q}_{t+1}^{\text{TOP}} \in \Delta_M$

Aggregate the final meta weights $\boldsymbol{p}_{t+1} \in \Delta_N$ via Eq. (3.12)

end for

problems. At a high level, online games can be regarded as a special instance of adaptive online learning. The goal is to ensure robust performance on hard problems (e.g., when facing a dishonest opponent) while achieving superior performance on easy problems (e.g., when the opponent is honest). In particular, the adaptivity (e.g., gradient variation) can yield faster-rate convergence as a direct consequence of the RVU property.

Since the faster-rate convergence requires the RVU property, in this part, we validate the effectiveness of our proposed UniGrad++.Correct in a simple two-player zero-sum game as an illustrating example. The game can be formulated as a min-max optimization problem of $\min_{\mathbf{x}\in\mathcal{X}} \max_{\mathbf{y}\in\mathcal{Y}} f(\mathbf{x}, \mathbf{y})$, in which the $\mathbf{x}$-player aims to minimize and the $\mathbf{y}$-player aims to maximize the objective. To validate the universality of our method, we consider the case that the game is either *bilinear*, i.e., $f(\mathbf{x}, \mathbf{y}) = \mathbf{x}^\top A\mathbf{y}$, or *strongly-convex-strongly-concave*, i.e., $f(\mathbf{x}, \mathbf{y})$ is λ-strongly convex in $\mathbf{x}$ and λ-strongly concave in $\mathbf{y}$. We denote the two players' gradients by $\mathbf{g}_t^{\mathbf{x}} = \nabla_{\mathbf{x}} f(\mathbf{x}, \mathbf{y})$ and $\mathbf{g}_t^{\mathbf{y}} = \nabla_{\mathbf{y}} f(\mathbf{x}, \mathbf{y})$. Besides, to validate the RVU property of our method, we investigate the players can be either honest or dishonest. We proceed under the following standard assumptions concerning the strategy domains and the gradients of the two players, following previous works (Farina et al., 2022). The second assumption, known as the smoothness assumption, is classical in online games.

Assumption 6. We make the following assumptions:

(i) The $\mathbf{x}$-player's strategy set is the simplex $\Delta_{d_{\mathbf{x}}}$, and the $\mathbf{y}$-player's strategy set is the simplex $\Delta_{d_{\mathbf{y}}}$. Moreover, the gradients of both players are uniformly bounded by G, i.e., $\|\mathbf{g}_t^{\mathbf{x}}\| \le G$ and $\|\mathbf{g}_t^{\mathbf{y}}\| \le G$ for all $t \in [T]$.

(ii) For $t \in [T]$, both players' gradients satisfy $\max\{\|\mathbf{g}_t^{\mathbf{x}} - \mathbf{g}_{t-1}^{\mathbf{x}}\|, \|\mathbf{g}_t^{\mathbf{y}} - \mathbf{g}_{t-1}^{\mathbf{y}}\|\} \le \|\mathbf{x}_t - \mathbf{x}_{t-1}\| + \|\mathbf{y}_t - \mathbf{y}_{t-1}\|$.

In Algorithm 6, we present the online game variant of UniGrad++.Correct for the $\mathbf{x}$-player, which ensures regret summation guarantees in the honest case and individual regret guarantees in the dishonest case, *without* requiring prior knowledge of the game type. Compared with the algorithm for the single-player setup, Algorithm 6 leverages additional problem structures for effective learning. Specifically, $\mathbf{x}$-player uses $\mathbf{g}_t^{\mathbf{x}}$ instead of the general

Table 3: Comparisons of our results with existing ones. In the honest case, the results are measured by the summation of all players' regret and in the dishonest case, the results are in terms of the individual regret of each player. Bilinear and strongly-convex-strongly-concave games are considered inside each case. $\star$ denotes the best result in each case (row).

	Games	Syrgkanis et al. (2015)	Zhang et al. (2022a)	**Ours**
Honest	bilinear	$\mathcal{O}(1)^\star$	$\mathcal{O}(\sqrt{T})$	$\mathcal{O}(1)^\star$
	strongly-convex strongly-concave	$\mathcal{O}(1)^\star$	$\mathcal{O}(\log T)$	$\mathcal{O}(1)^\star$
Dishonest	bilinear	$\mathcal{O}(\sqrt{T})^\star$	$\mathcal{O}(\sqrt{T})^\star$	$\mathcal{O}(\sqrt{T \log T})^\star$
	strongly-convex strongly-concave	$\mathcal{O}(\sqrt{T})$	$\mathcal{O}(\log T)^\star$	$\mathcal{O}(\log T)^\star$

$\nabla f_t(\mathbf{x}_t)$ as the feedback. The analogous variant for the $\mathbf{y}$-player follows the same design and is omitted for brevity. Here, the total number of base learners for both the $\mathbf{x}$-player and the $\mathbf{y}$-player is $N = 1 + |\mathcal{H}^{\text{sc}}| = \mathcal{O}(\log T)$. We conclude our results in Theorem 7 and defer the proof to Appendix D.5.

Theorem 7. *Under Assumption 6, by setting*

$$C_0 = \max\left\{1, 8D, 4\gamma^{\text{TOP}}, 16C_{31}, 16C_{32}, 4\gamma_{\mathbf{x}}^{\text{TOP}}, 4\gamma_{\mathbf{y}}^{\text{TOP}}\right\},$$
$$\gamma_{\mathbf{x}}^{\text{MID}} = \gamma_{\mathbf{y}}^{\text{MID}} = 128 + 128G^2 + 40\sqrt{2}, \quad \gamma_{\mathbf{x}}^{\text{TOP}} = \gamma_{\mathbf{y}}^{\text{TOP}} = 512 + 512G^2 + 160\sqrt{2},$$

where

$$C_{31} = \frac{32 + 64G^2}{Z^{\mathbf{x}}} + \frac{32}{Z^{\mathbf{y}}} + 20\sqrt{2}, \quad C_{32} = \frac{32 + 64G^2}{Z^{\mathbf{y}}} + \frac{32}{Z^{\mathbf{x}}} + 20\sqrt{2},$$
$$Z^{\mathbf{x}} = \max\{GD + \gamma_{\mathbf{x}}^{\text{MID}} D^2, 1 + \gamma_{\mathbf{x}}^{\text{MID}} D^2 + 2\gamma_{\mathbf{x}}^{\text{TOP}}\},$$
$$Z^{\mathbf{y}} = \max\{GD + \gamma_{\mathbf{y}}^{\text{MID}} D^2, 1 + \gamma_{\mathbf{y}}^{\text{MID}} D^2 + 2\gamma_{\mathbf{y}}^{\text{TOP}}\},$$

for bilinear and strongly-convex-strongly-concave games, Algorithm 6 enjoys $\mathcal{O}(1)$ regret summation in the honest case, and achieves $\mathcal{O}(\sqrt{T \log T})$ and $\mathcal{O}(\log T)$ regret bounds for bilinear and strongly-convex-strongly-concave games respectively in the dishonest case.

Table 3 compares our approach with Rakhlin and Sridharan (2013b) and Zhang et al. (2022a). Specifically, ours is better than Rakhlin and Sridharan (2013b) in the strongly-convex-strongly-concave games in the dishonest case due to its universality, and better than Zhang et al. (2022a) in the honest case since our approach enjoys gradient-variation bounds that are essential in achieving fast rates for regret summation.

6.4 Extension to Anytime Setting: Dynamic Online Ensemble

We start by observing that previous methods require the knowledge of the time horizon T in advance, which is often unavailable in practice. To this end, we propose a *dynamic online ensemble* — an anytime framework that avoids dependence on the time horizon T and adjusts its candidate pools dynamically during the online learning process.

As established in Section 2.2, all base learners are OOMD-type algorithms with adaptive step sizes, which means the base learners are inherently anytime given the choice of the curvature parameter. Therefore, we consider the dependence on the time horizon T within the meta algorithm design as well as the scheduling of the curvature parameter.

Scheduling. To begin with, to make the curvature parameter scheduling suitable for the anytime setting, we consider maintaining *infinite* candidates of the curvature coefficient and only activating a finite set of them at each round. This idea is inspired by Mhammedi et al. (2019). Specifically, we define two candidate coefficient pools for the exp-concave and strongly convex cases, respectively:

$$\mathcal{H}^{\exp} \triangleq \{\alpha_i = 2^{-i}, \text{ for } i \in \{0\} \cup \mathbb{N}\}, \quad \mathcal{H}^{\text{sc}} \triangleq \{\lambda_i = 2^{-i}, \text{ for } i \in \{0\} \cup \mathbb{N}\}. \tag{6.8}$$

Intuitively, if the time horizon T is known, we only need to discretize the possible range of $[1/T, 1]$ to get the candidate pool, as in Eq. (2.5). In contrast, when T is unknown, it may grow arbitrarily large, causing the lower bound of the feasible range to converge to zero. Therefore, we deploy 2^{-i} for all $i \in \{0\} \cup \mathbb{N}$ as all possible candidates of the curvature coefficient. For convex functions, we still maintain a single base learner $\mathcal{B}^{\text{c}}$ as there is no unknown-curvature-coefficient issues.

Furthermore, as we cannot implement infinite base learners in the actual running of the algorithm, we define two *active* versions (denoted by $\mathcal{H}_t^{\exp}$ and $\mathcal{H}_t^{\text{sc}}$) of them which indicates that when $\lambda_i \in \mathcal{H}_t^{\text{sc}}$, the i-th base learner is active. Formally,

$$\begin{aligned} \mathcal{H}_t^{\text{sc}} &\triangleq \left\{\lambda_i = 2^{-i} \text{ and } t \geq \left(s_i^{\text{sc}} \triangleq \frac{1}{\lambda_i}\right), \text{ for } i \in \{0\} \cup \mathbb{N}\right\}, \\ \mathcal{H}_t^{\exp} &\triangleq \left\{\alpha_i = 2^{-i} \text{ and } t \geq \left(s_i^{\exp} \triangleq \frac{1}{\alpha_i}\right), \text{ for } i \in \{0\} \cup \mathbb{N}\right\}. \end{aligned} \tag{6.9}$$

We denote their sizes by $N_t^{\text{sc}} = |\mathcal{H}_t^{\text{sc}}|$ and $N_t^{\exp} = |\mathcal{H}_t^{\exp}|$. This means that the base learner with α_i is only activated from $t = s_i^{\exp}$. For example, the base learner with $\alpha_i = \frac{1}{8}$ is activated from $t = s_i^{\exp} = 8$.

Meta Algorithm. Subsequently, we consider making the meta algorithm anytime. Note that when the aforementioned *infinitely many* base learners, the meta algorithms such as MoM (used in UniGrad.Correct and its one-gradient version) and Optimistic-Adapt-ML-Prod (used in UniGrad++.Bregman and its one-gradient version) are *not* applicable because they only support a fixed and number of base learners.

Fortunately, Xie et al. (2024) proposed a variant of Optimistic-Adapt-ML-Prod that can handle the case of infinite base learners, which is perfectly suitable for our anytime setting. Specifically, let $\mathcal{A}_t$ be the set of active experts at round t and let $N_t = |\mathcal{A}_t|$ denote its size. We initialize $\mathcal{A}_0 = \{\mathcal{B}^{\text{c}}, \mathcal{B}_0^{\text{sc}}, \mathcal{B}_0^{\exp}\}$, where $\mathcal{B}_0^{\text{sc}}$ and $\mathcal{B}_0^{\exp}$ are associated with the initial coefficients λ_0 and α_0, respectively. At t-th round, a newly added expert i is initialized with weight $W_{t,i} = 1$ and learning rate $\varepsilon_{t,i} = \frac{1}{\sqrt{5}}$. The meta algorithm submits $\boldsymbol{p}_t \in \Delta_{N_t}$ as

$$p_{t,i} = \frac{\varepsilon_{t,i} W_{t,i} \exp(\varepsilon_{t,i} m_{t,i})}{\sum_{j \in [N_t]} \varepsilon_{t,j} W_{t,j} \exp(\varepsilon_{t,j} m_{t,j})}, \text{ for all } i \in [N_t]. \tag{6.10}$$

Algorithm 7 Anytime Variant of UniGrad++.Bregman

Initialize: $\mathcal{M}$ — meta learner Optimistic-Adapt-ML-Prod variant
$\{\mathcal{B}_i\}$ — base learners as specified in Section 2.3 (new scheduled in Eq. (6.8))
$\mathcal{A}_0 = \emptyset$ — active set
for $t = 1$ **to** T **do**
  Activate the base learners with $\lambda_i \in \mathcal{H}_t^{\text{sc}}$ or $\alpha_i \in \mathcal{H}_t^{\exp}$, initialize weights and learning rates as $W_{t,i} = 1$ and $\varepsilon_{t,i} = \frac{1}{\sqrt{5}}$, and add them to $\mathcal{A}_{t-1}$ to obtain $\mathcal{A}_t$
  Receive $\{\mathbf{x}_{t,i}\}_{i\in[N_t]}$ from $\{\mathcal{B}_i\}_{i\in[N_t]}$ and $\boldsymbol{p}_t \in \Delta_{N_t}$ from $\mathcal{M}$ via Eq. (6.10)
  Submit $\mathbf{x}_t = \sum_{i\in[N_t]} p_{t,i}\mathbf{x}_{t,i}$, suffer $f_t(\mathbf{x}_t)$, and observe $\nabla f_t(\mathbf{x}_t)$
  $\{\mathcal{B}_i^{\text{sc}}\}_{i\in[N_t^{\text{sc}}]}$, $\{\mathcal{B}_i^{\exp}\}_{i\in[N_t^{\exp}]}$ and $\mathcal{B}^{\text{c}}$ update their own decisions to $\{\mathbf{x}_{t+1,i}\}_{i\in[N_t]}$ using surrogate losses of $\{h_{t,i}^{\text{sc}}(\cdot)\}_{\lambda_i\in\mathcal{H}_t^{\text{sc}}}$ (5.1), $\{h_{t,i}^{\exp}(\cdot)\}_{\alpha_i\in\mathcal{H}_t^{\exp}}$ (5.2), and $h_t^{\text{c}}(\cdot)$ (5.2)
  Calculate $\boldsymbol{m}_{t+1}$ (6.11) and $\boldsymbol{r}_t$ using $\{\mathbf{x}_{t,i}\}_{i=1}^{N_t}$, $\mathbf{x}_t$, $\nabla f_t(\mathbf{x}_t)$, and $\{\mathbf{x}_{t+1,i}\}_{i=1}^{N_t}$, send them to $\mathcal{M}$, and obtain $(W_{t+1,1}, \ldots, W_{t+1,N_t})$ via Eq. (6.13)
end for

The optimistic vector $\boldsymbol{m}_t \in \mathbb{R}^{N_t}$ is designed as

$$m_{t,i} = \langle \nabla f_{t-1}(\mathbf{x}_{t-1}), \mathbf{x}_t - \mathbf{x}_{t,i}\rangle \text{ for } i = 1, \text{ and } m_{t,i} = 0 \text{ for } i > 1. \tag{6.11}$$

After receiving the loss vector $\boldsymbol{r}_t = (r_{t,1}, \ldots, r_{t,N_t}) \in \mathbb{R}^{N_t}$, where $r_{t,i} \triangleq \langle \boldsymbol{\ell}_t, \boldsymbol{p}_t - \boldsymbol{e}_i\rangle$ and $\boldsymbol{\ell}_{t,i} = \langle \nabla f_t(\mathbf{x}_t), \mathbf{x}_{t,i}\rangle$, for $i \in [N_t]$, it chooses the learning rate as

$$\varepsilon_{t+1,i} = \sqrt{\frac{1}{5 + \sum_{s=s_i^{\{\exp,\text{sc}\}}}^{t} (r_{s,i} - m_{s,i})^2}}, \text{ for all } i \in [N_t]. \tag{6.12}$$

Finally, for each $i \in [N_t]$, the meta algorithm updates the weights as

$$W_{t+1,i} = \left(W_{t,i} \exp\left(\varepsilon_{t,i} r_{t,i} - \varepsilon_{t,i}^2 (r_{t,i} - m_{t,i})^2\right)\right)^{\frac{\varepsilon_{t+1,i}}{\varepsilon_{t,i}}}, \text{ for all } i \in [N_t]. \tag{6.13}$$

The corresponding guarantee is deferred to Lemma 9 in Appendix D.6.

Note that the meta algorithm required here does not enjoy stability-induced negative terms as it falls in the Adapt-ML-Prod family. Therefore, this extension cannot be applied to UniGrad.Correct and UniGrad++.Correct because correction-based methods require stability-induced negative terms for effective cancellations. On the contrary, UniGrad.Bregman and UniGrad++.Bregman can be made anytime by replacing their original meta algorithm, i.e., Optimistic-Adapt-ML-Prod, with this anytime variant in a straightforward manner algorithmically. For simplicity, we only present the anytime extension of UniGrad++.Bregman here, where the algorithm is concluded in Algorithm 7 and the corresponding guarantee is presented in Theorem 8. The proof is deferred to Appendix D.6.

Theorem 8. *Under Assumptions 1, 2, 5, and without the knowledge of time horizon T, by setting the learning rate of meta algorithm as Eq. (6.12),* UniGrad++.Bregman *in Algorithm 5 achieves the following anytime universal gradient-variation regret guarantees using*

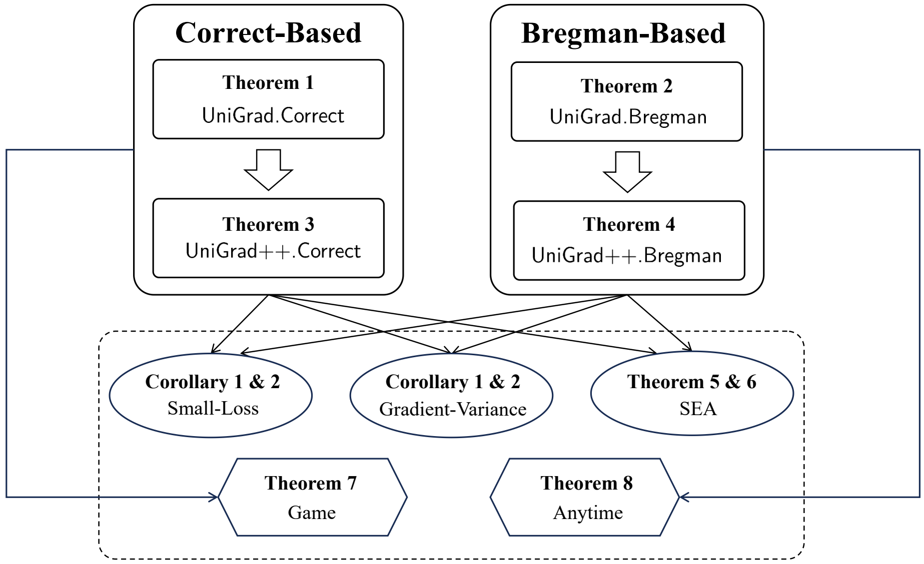


Figure 2: The summary of the theoretical results in our work. Specifically, we propose two methods named UniGrad.Correct and UniGrad.Bregman (Theorem 1 and Theorem 2) to achieve gradient-variation universal regret. Both methods can be strengthened to the one-gradient feedback scenario (Theorem 3 and Theorem 4). Besides, our results find important implications in small-loss and gradient-variance problem-dependent regret (Corollary 1 and Corollary 2), stochastically extended adversarial (SEA) model (Theorem 5 and Theorem 6), and game theory (Theorem 7). Furthermore, our results can be extended to the anytime setup (without knowing the time horizon T) in Theorem 8.

only one gradient per round: for any $\tau \in [T]$, we have

$$\textsc{Reg}_\tau \triangleq \sum_{t=1}^{\tau} f_t(\mathbf{x}_t) - \min_{\mathbf{x}\in\mathcal{X}} \sum_{t=1}^{\tau} f_\tau(\mathbf{x}) \leq \begin{cases} \mathcal{O}\left(\frac{1}{\lambda}\log V_\tau\right), & \textit{when } \{f_t\}_{t=1}^T \textit{ are } \lambda\textit{-strongly convex},\\ \mathcal{O}\left(\frac{d}{\alpha}\log V_\tau\right), & \textit{when } \{f_t\}_{t=1}^T \textit{ are } \alpha\textit{-exp-concave},\\ \mathcal{O}(\sqrt{V_\tau}), & \textit{when } \{f_t\}_{t=1}^T \textit{ are convex.}\end{cases}$$

7. Discussions

This section provides further discussions to better understand our proposed methods, which is organized along two directions: *(i)* a comprehensive comparison between our two proposed methods (UniGrad.Correct and UniGrad.Bregman; as well as their respective one-gradient variants), and *(ii)* a detailed technical comparison of UniGrad.Correct with its previous conference version (Yan et al., 2023).

7.1 Comparison of UniGrad.Correct and UniGrad.Bregman

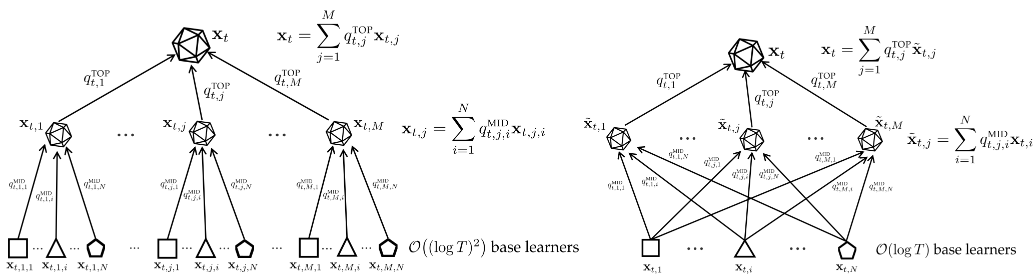
To achieve the desired gradient-variation universal regret in online learning, we propose two different methods called UniGrad.Correct and UniGrad.Bregman, each with its own merits and characteristics. Here we provide a systematic comparison along several key dimensions.

Table 4: Comparison of the algorithmic structures of different methods. UniGrad.Correct and UniGrad.Bregman are two different methods to achieve the gradient-variation universal regret, and UniGrad++.Correct and UniGrad++.Bregman are their one-gradient variants.

Method	**Meta Algorithm**	**Base Algorithm**	**Remark**
UniGrad.Correct	MoM (2-layer)	OOMD with $\{\nabla f_t(\mathbf{x}_{t,i})\}_{i=1}^N$	three layers; $N = \Theta(\log T)$ base learners, N gradient queries per round
UniGrad++.Correct	MoM (2-layer)	OOMD with $\nabla f_t(\mathbf{x}_t)$	three layers; $N = \Theta(\log T)$ base learners, 1 gradient query per round
UniGrad.Bregman	Optimistic-Adapt-ML-Prod	OOMD with $\{\nabla f_t(\mathbf{x}_{t,i})\}_{i=1}^N$	two layers; $N = \Theta(\log T)$ base learners, N gradient queries per round
UniGrad++.Bregman	Optimistic-Adapt-ML-Prod	OOMD with $\nabla f_t(\mathbf{x}_t)$	two layers; $N = \Theta(\log T)$ base learners, 1 gradient query per round

Overview of Results. The theoretical contributions of this paper can be organized into two distinct methodological approaches. As demonstrated in Figure 2, we introduce two principal algorithms: UniGrad.Correct and UniGrad.Bregman, both of which achieve gradient-variation universal regret bounds (see Theorem 1 and Theorem 2). Each method can be further extended to the one-gradient feedback setting (Theorem 3 and Theorem 4), requiring only a single gradient query per round. Beyond these core regret guarantees, both methods support a range of important implications and applications. These include problem-dependent bounds for small-loss and gradient-variance settings (Corollary 1 and Corollary 2), adaptive guarantees for the SEA model (Theorem 5 and Theorem 6). UniGrad.Correct is particularly well-suited for faster convergence in online games as it can preserve the its RVU property (Theorem 7). In contrast, UniGrad.Bregman features a simpler structure and is more amenable to extension to the anytime setting, where the time horizon T is unknown in advance (Theorem 8).

Regret Bounds and Algorithmic Structures. While both methods achieve gradient-variation universal regret, they differ in the regret bound for convex functions. UniGrad.Correct provides an $\mathcal{O}(\sqrt{V_T \log V_T})$ regret bound for convex functions, while UniGrad.Bregman can achieve the optimal $\mathcal{O}(\sqrt{V_T})$ bound for convex functions, matching the lower bound established in Chiang et al. (2012). A concrete comparison of the regret bounds can be found in Table 1. The most fundamental difference lies in algorithmic structures due to their different methodologies. As summarized in Table 4, UniGrad.Correct employs a three-layer online ensemble with $N = \Theta(\log T)$ base learners, while UniGrad.Bregman uses a more streamlined two-layer structure with the same number of base learners. For both UniGrad.Correct and UniGrad.Bregman, they require N gradient queries per round, and their one-gradient variants successfully reduce the number of gradient queries to 1 per round.



(a) Method of Yan et al. (2023) (b) UniGrad.Correct

Figure 3: Comparison of the three-layer online ensemble structures between the conference version (Yan et al., 2023) and UniGrad.Correct. The key difference lies in how base learners are managed: Yan et al. (2023) maintain a separate group of base learners for each MoM-Mid, whereas UniGrad.Correct employs *shared* base learners across all MoM-Mid's, thereby reducing the total number of base learners from $\mathcal{O}((\log T)^2)$ to $\mathcal{O}(\log T)$.

Technical Differences. The two methods differ significantly in their core technical approaches, particularly in how they convert the empirical gradient variation term $\bar{V}_T$ to the gradient variation term V_T (see Lemma 1 and Lemma 6).

- **UniGrad.Correct:** The most important technical feature of UniGrad.Correct is the *cancellation argument* based on the (positive and negative) stability term and the curvature-induced negative term. By carefully exploiting these negative terms together with the cascade correction scheme, UniGrad.Correct attains the desired universal regret with a three-layer online ensemble structure. The development significantly advances the adaptivity of the online ensemble framework, providing a principled basis for analyzing the stability of multi-layer online ensembles.
- **UniGrad.Bregman:** The key innovation of UniGrad.Bregman is to eliminate the meta level stability term when converting empirical gradient variation to the desired gradient variation (see Lemma 6). This is achieved by extracting the negative Bregman divergence arising from the linearization of the regret from the beginning. This mechanism greatly simplifies the algorithmic analysis and design and provides a useful tool for future research in adaptive online learning.

7.2 Comparison of UniGrad.Correct with Conference Version

The design of UniGrad.Correct builds upon and significantly improves the correction-based method presented in our conference version (Yan et al., 2023). While both share the fundamental idea of using correction terms to handle positive stability terms, our current methods achieve superior theoretical guarantees and computational efficiency.

Conference Version. The algorithm of Yan et al. (2023) employs a three-layer online ensemble, with the final output at each round computed as:

$$\mathbf{x}_t = \sum_{j=1}^{M} q_{t,j}^{\text{TOP}} \mathbf{x}_{t,j}, \quad \mathbf{x}_{t,j} = \sum_{i=1}^{N} q_{t,j,i}^{\text{MID}} \mathbf{x}_{t,j,i}.$$

Here, $\boldsymbol{q}_t^{\text{TOP}}$ is the decision of MoM-Top, which connects with $M = \mathcal{O}(\log T)$ MoM-Mid. Similarly, $\boldsymbol{q}_{t,j}^{\text{MID}}$ denotes the decisions of MoM-Mid, which further connects with $N = \mathcal{O}(\log T)$ base learners. As a result, the algorithm requires maintaining $\mathcal{O}(MN) = \mathcal{O}((\log T)^2)$ base learners in total, as illustrated in Figure 3(a). To handle the positive stability term $\|\mathbf{x}_t - \mathbf{x}_{t-1}\|^2$, Yan et al. (2023) leveraged the following *cascaded* stability decomposition:

$$\|\mathbf{x}_t - \mathbf{x}_{t-1}\|^2 \lesssim \|\boldsymbol{q}_t^{\text{TOP}} - \boldsymbol{q}_{t-1}^{\text{TOP}}\|_1^2 + \sum_{j=1}^{M} q_{t,j}^{\text{TOP}} \|\mathbf{x}_{t,j} - \mathbf{x}_{t-1,j}\|^2, \tag{7.1}$$

$$\|\mathbf{x}_{t,j^\star} - \mathbf{x}_{t-1,j^\star}\|^2 \lesssim \|\boldsymbol{q}_{t,j^\star}^{\text{MID}} - \boldsymbol{q}_{t-1,j^\star}^{\text{MID}}\|_1^2 + \sum_{i=1}^{N} q_{t,j^\star,i}^{\text{MID}} \|\mathbf{x}_{t,j^\star,i} - \mathbf{x}_{t-1,j^\star,i}\|^2. \tag{7.2}$$

According to the second term in the right-hand side of Eq. (7.1), the top-layer correction term is set as $q_{t,j}^{\text{TOP}}\|\mathbf{x}_{t,j} - \mathbf{x}_{t-1,j}\|^2$, which generates additional positive term $\|\mathbf{x}_{t,j^\star} - \mathbf{x}_{t-1,j^\star}\|^2$. This term is further decomposed using Eq. (7.2), whose second term in the right-hand side motives the injection of the middle-layer correction term $q_{t,j^\star,i}^{\text{MID}}\|\mathbf{x}_{t,j^\star,i} - \mathbf{x}_{t-1,j^\star,i}\|^2$ to $j^\star$-th MoM-Mid to ensure a property cancellation. While this three-layer ensemble and correction scheme is intuitive and conceptually straightforward, it has a limitation: each MoM-Mid in the middle layer requires maintaining its own set of base learners, leading to a total complexity of $\mathcal{O}(\log T) \times \mathcal{O}(\log T) = \mathcal{O}((\log T)^2)$ base learners. This design introduces redundancy and unnecessary complexity.

Improved Version in Current Paper. UniGrad.Correct reduces the number of base learners to $\mathcal{O}(\log T)$ by carefully restructuring the framework, though the three-layer structure and cascade corrections are still necessary. As shown in Figure 3(b), the proposed UniGrad.Correct algorithm produces the final output at each round as:

$$\mathbf{x}_t = \sum_{i=1}^{N} p_{t,i}\mathbf{x}_{t,i}, \quad \boldsymbol{p}_t = \sum_{j=1}^{M} q_{t,j}^{\text{TOP}} \boldsymbol{q}_{t,j}^{\text{MID}},$$

where $\{\mathbf{x}_{t,i}\}_{i=1}^N$ are the local decisions returned by the base learners, with $N = \mathcal{O}(\log T)$. The meta combination weight $p_{t,i}$ is calculated based on a two-layer MsMwC. In fact, this update can be equivalently understood as follows:

$$\mathbf{x}_t = \sum_{i=1}^{N} p_{t,i}\mathbf{x}_{t,i} = \sum_{i=1}^{N} \Big(\sum_{j=1}^{M} q_{t,j}^{\text{TOP}} q_{t,j,i}^{\text{MID}}\Big)\mathbf{x}_{t,i} = \sum_{j=1}^{M} q_{t,j}^{\text{TOP}} \Big(\sum_{i=1}^{N} q_{t,j,i}^{\text{MID}}\mathbf{x}_{t,i}\Big) \triangleq \sum_{j=1}^{M} q_{t,j}^{\text{TOP}} \widetilde{\mathbf{x}}_{t,j}, \tag{7.3}$$

where the last equality defines new hidden aggregation nodes $\widetilde{\mathbf{x}}_{t,j} \triangleq \sum_{i=1}^N q_{t,j,i}^{\text{MID}}\mathbf{x}_{t,i}$. Compared with the aggregation of $\mathbf{x}_t = \sum_{j=1}^M q_{t,j}^{\text{TOP}}\mathbf{x}_{t,j}$ and $\mathbf{x}_{t,j} = \sum_{i=1}^N q_{t,j,i}^{\text{MID}}\mathbf{x}_{t,j,i}$ in the conference version (Yan et al., 2023), it can be seen that hidden aggregations $\{\widetilde{\mathbf{x}}_{t,j}\}_{j=1}^M$ use the *shared* base learner group, namely, $\mathbf{x}_{t,j,i} = \mathbf{x}_{t,i}$ for any $j \in [M]$. This "base learner sharing" design is the key improvement over the conference version: by requiring the middle-layer meta algorithm to use a shared set of base learners across all MoM-Mid, UniGrad.Correct reduces the number of base learners from $\mathcal{O}((\log T)^2)$ to $\mathcal{O}(\log T)$, significantly enhancing efficiency.

The new understanding in Eq. (7.3) can also benefit and simplify the stability analysis for UniGrad.Correct in a similar cascade way to the conference version (Yan et al., 2023):

$$\|\mathbf{x}_t - \mathbf{x}_{t-1}\|^2 \lesssim \|\boldsymbol{q}_t^{\text{TOP}} - \boldsymbol{q}_{t-1}^{\text{TOP}}\|_1^2 + \sum_{j=1}^{M} q_{t,j}^{\text{TOP}} \|\widetilde{\mathbf{x}}_{t,j} - \widetilde{\mathbf{x}}_{t-1,j}\|^2,$$

$$\|\widetilde{\mathbf{x}}_{t,j^\star} - \widetilde{\mathbf{x}}_{t-1,j^\star}\|^2 \lesssim \|\boldsymbol{q}_{t,j^\star}^{\text{MID}} - \boldsymbol{q}_{t-1,j^\star}^{\text{MID}}\|_1^2 + \sum_{i=1}^{N} q_{t,j^\star,i}^{\text{MID}} \|\mathbf{x}_{t,i} - \mathbf{x}_{t-1,i}\|^2.$$

This decomposition not only provides a conceptually more straightforward proof for Lemma 5, but also establishes a principled framework for analyzing even deeper online ensemble structures. We believe these insights could be of interest to the community.

8. Experiments

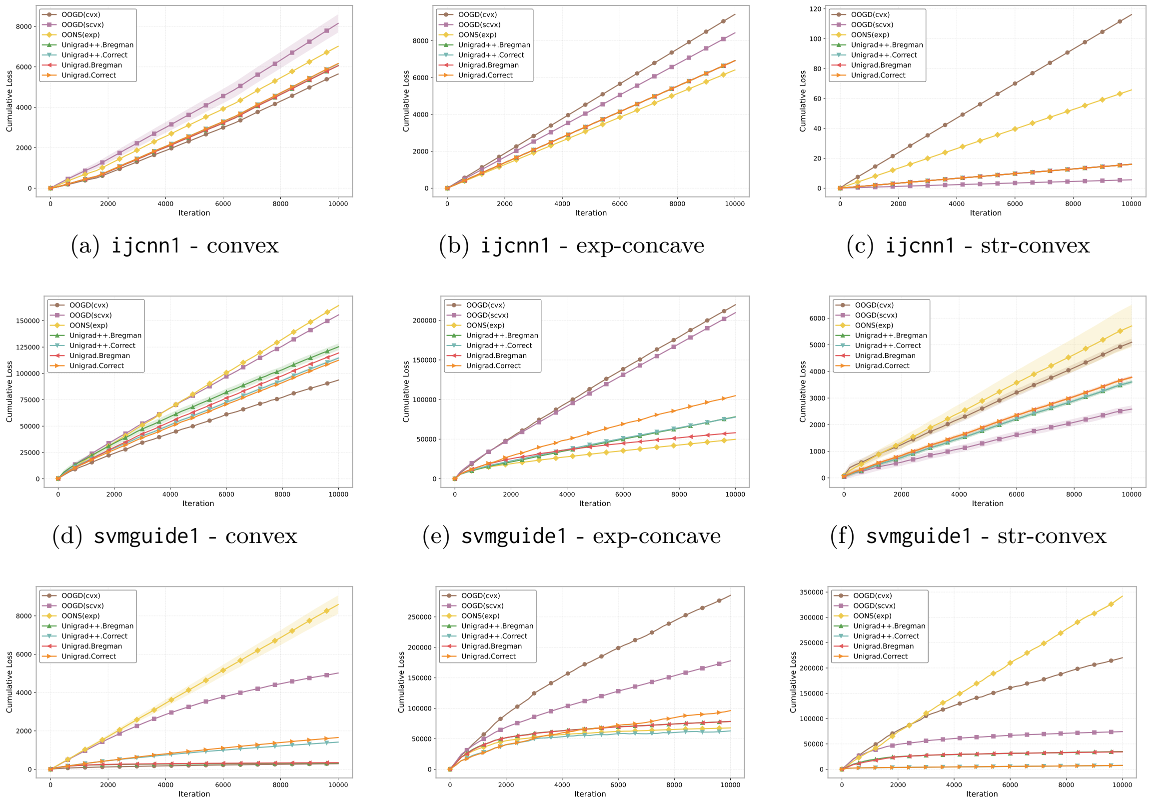
This section provides empirical studies to validate the effectiveness of our algorithms. Through empirical evaluations, we aim to answer the following three questions:

- **Universality**: Can our methods automatically adapt to the unknown curvature of online functions and achieve comparable performance with the optimal algorithm specifically designed for each problem instance?
- **Adaptivity**: Can our methods adapt to the gradient variation V_T and achieve better performance than the methods that are not fully gradient-variation adaptive, e.g., the method of Zhang et al. (2022a), when V_T is small?
- **Efficiency**: Can our one-gradient improvements UniGrad++ (Correct/Bregman) achieve comparable performance to their vanilla versions UniGrad (Correct/Bregman) while with significantly reduced gradient query cost?

Contenders and Configurations. To validate the universality, we compare our methods with the optimal algorithm specifically designed for each problem instance, as specified in Section 2.3. For the adaptivity validation, we compare our methods with the USC algorithm (Zhang et al., 2022a), which enjoys the universal regret of $\mathcal{O}(\sqrt{T})$ for convex functions, $\mathcal{O}(\frac{d}{\alpha}\log V_T)$ for exp-concave functions, and $\mathcal{O}(\frac{1}{\lambda}\log V_T)$ for strongly convex functions, respectively. Finally, for efficiency validation, we compare the one-gradient methods (UniGrad++.Correct and UniGrad++.Bregman) with their vanilla versions (UniGrad.Correct and UniGrad.Bregman), which require $\mathcal{O}(\log T)$ gradient queries per round.

In all experiments, we set the total time horizon to $T = 10{,}000$ and choose the decision domain $\mathcal{X}$ as the unit ball. All hyper-parameters are set to be theoretically optimal.

- All algorithm hyper-parameters are set to be theoretically optimal. To validate the universality of our approach, we conduct experiments on three datasets from the LIBSVM repository (Chang and Lin, 2011): `ijcnn1`, `svmguide1`, and `skin_nonskin`. All of these are binary classification datasets, and we transform the labels to $\{-1, 1\}$. At each round $t \in [T]$, we randomly sample a data point $(\mathbf{a}_t, y_t)$ from the chosen dataset to construct the loss function $f_t(\mathbf{x})$. Specifically, for the convex setting, we choose the linear loss



(a) ijcnn1 - convex (b) ijcnn1 - exp-concave (c) ijcnn1 - str-convex

(d) svmguide1 - convex (e) svmguide1 - exp-concave (f) svmguide1 - str-convex

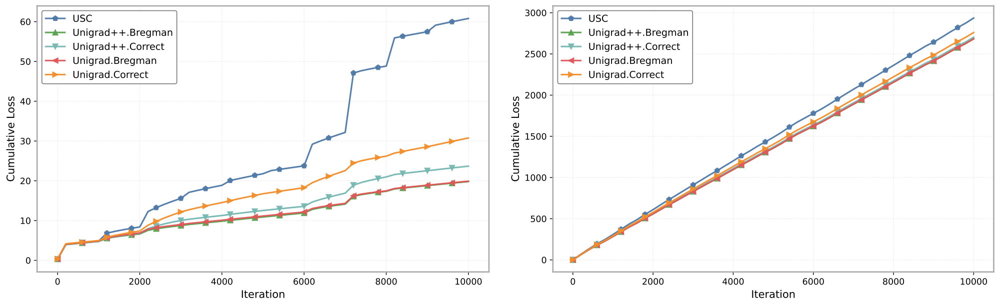
(g) skin_nonskin - convex (h) skin_nonskin - exp-concave (i) skin_nonskin - str-convex

Figure 4: **Universality**: comparisons on three problem classes—convex, exp-concave, and strongly convex—across three datasets (ijcnn1, svmguide1, skin_nonskin). Rows correspond to datasets, columns correspond to problem classes. Our methods are evaluated against the optimal algorithm specifically designed for each class, showing comparable regret performance.

function $f_t(\mathbf{x}) = \max(0, 1 - y_t \cdot \mathbf{a}_t^\top \mathbf{x})$. For the exp-concave setting, we use the logistic loss function $f_t(\mathbf{x}) = \log(1 + \exp(-y_t \cdot \mathbf{a}_t^\top \mathbf{x}))$. For the strongly convex setting, we choose the loss function $f_t(\mathbf{x}) = \max(0, 1 - y_t \cdot \mathbf{a}_t^\top \mathbf{x}) + \frac{1}{2}\|\mathbf{x}\|_2^2$.

- To validate the adaptivity of our approach, we first compare our method with USC (Zhang et al., 2022a) on the ijcnn1 dataset, where the gradient variation satisfies $V_T = \mathcal{O}(T)$. We then construct an online function sequence with $V_T = \mathcal{O}(1)$ and perform the same comparison on this sequence. Specifically, we choose the loss functions as $f_t(\mathbf{x}) = \mathbf{a}_{i_t}^\top \mathbf{x} + b_{i_t}$, where $\mathbf{a}_0 = [0.2, 0.2]^\top$, $b_0 = 0$, and $i_t = \lfloor 10t/T \rfloor$. The parameters evolve gradually as $\mathbf{a}_i = \mathbf{a}_{i-1} + 0.1 \cdot \varepsilon_i$ and $b_i = b_{i-1} + 0.1 \cdot \xi_i$ for $i \in [10]$, with noise $\varepsilon_i \sim \mathcal{N}(\mathbf{0}, I_2)$ and $\xi_i \sim \mathcal{N}(0, 1)$, such that the total gradient variation of the online functions sequence can be treated as a constant.

- For efficiency evaluation, we compare the total running time of the one-gradient variants UniGrad++.(Correct/Bregman) with their vanilla counterparts UniGrad.(Correct/Bregman) across all three datasets (ijcnn1, svmguide1, and skin_nonskin) in the exp-concave case.



(a) $V_T = \mathcal{O}(1)$ (b) $V_T = \mathcal{O}(T)$

Figure 5: **Adaptivity**: comparisons on the adaptivity of our methods against USC of Zhang et al. (2022a). Our methods outperform USC when the gradient variation V_T is small, e.g., $V_T = \mathcal{O}(1)$ in Figure 5(a), and show comparable performance when $V_T = \mathcal{O}(T)$ in Figure 5(b).

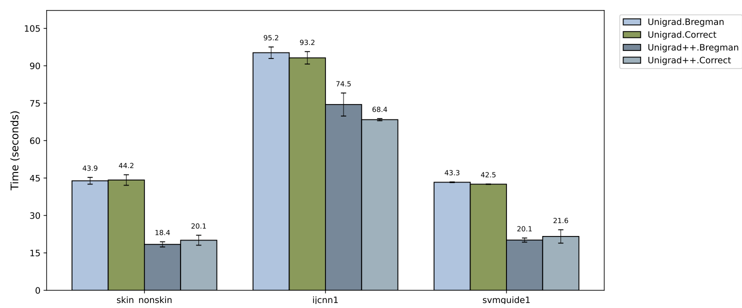


Figure 6: **Efficiency**: comparisons on the efficiency of our one-gradient improvements UniGrad++ (Correct/Bregman) against their vanilla versions UniGrad (Correct/Bregman). Our one-gradient improvements are more efficient than their vanilla versions in terms of the time complexity.

We report the average cumulative losses with standard deviations of 5 independent runs to obtain convincing results. Only the randomness of the initialization is preserved.

Numeric Results. Figure 4 shows the universality comparison results. Our methods are compared with the optimal algorithm specifically designed for each problem instance, validating the universality of our methods, and show comparable performance across different problem classes and datasets.

Figure 5 presents the adaptivity comparison results. Our methods outperform the USC algorithm (Zhang et al., 2022a) when the gradient variation V_T is small, e.g., $V_T = \mathcal{O}(1)$ in Figure 5(a), and show comparable performance when $V_T = \mathcal{O}(T)$ in Figure 5(b).

Figure 6 shows the efficiency comparison results. Our one-gradient variants are more efficient than their vanilla multi-gradient versions in time complexity, while maintaining comparable performance, as shown in Figure 4 and Figure 5.

9. Conclusion

In this paper, we addressed the fundamental challenge of achieving both universality and adaptivity in online learning by introducing UniGrad, a new approach that obtains gradient-variation universal regret guarantees. We proposed two distinct methods, UniGrad.Correct and UniGrad.Bregman, each with its own technical innovations. UniGrad.Correct employs a three-layer online ensemble with cascaded correction terms, achieving regret bounds of $\mathcal{O}(\frac{1}{\lambda}\log V_T)$ for λ-strongly convex functions, $\mathcal{O}(\frac{d}{\alpha}\log V_T)$ for α-exp-concave functions, and $\mathcal{O}(\sqrt{V_T \log V_T})$ for convex functions, while preserving the RVU property crucial for fast convergence in online games. In contrast, UniGrad.Bregman leverages a novel Bregman divergence analysis to achieve the same bounds for strongly convex and exp-concave functions, but improves upon the convex case with the optimal $\mathcal{O}(\sqrt{V_T})$ regret. Both methods maintain $\mathcal{O}(\log T)$ base learners and require $\mathcal{O}(\log T)$ gradient queries per round.

Building on these results, we further developed UniGrad++, which preserves the same regret guarantees while reducing the gradient query cost to just one per round via a surrogate optimization technique. We also extended our method to an anytime variant that removes the need to know the horizon T in advance, using a dynamic online ensemble framework that adjusts the number of base learners based on monitoring metrics. Importantly, our results lead to broader implications and applications, including optimal small-loss and gradient-variance bounds, novel guarantees for the stochastically extended adversarial model, and faster convergence in online games.

There are several interesting future directions worthy investigating. The first is to explore whether the computational overhead can be further reduced by requiring only 1 projection per round for gradient-variation universal regret (Zhao et al., 2022; Yang et al., 2024). A second direction is to extend our results to unconstrained domains in order to achieve parameter-free online learning (Cutkosky and Orabona, 2018), thereby broadening its applicability. Finally, current universal online learning methods assume a homogeneous setting where all online functions share the same curvature class (convex, λ-strongly convex, or α-exp-concave). A more challenging and realistic goal is to handle heterogeneous environments where the curvature may vary over time. One possible starting point is the recently proposed contaminated OCO setting (Kamijima and Ito, 2024), which assumes that the objective functions are mostly uniform but may be contaminated by a small fraction of rounds – up to some unknown k – where the curvature class differs.

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Contents

Appendix A. Omitted Proofs for Section 3

In this section, we provide the proofs for the results in Section 3, including Lemma 2, Lemma 3 and Lemma 4. For simplicity, we introduce the following notations denoting the stability of the final and intermediate decisions of the algorithm. Specifically, for any $j \in [M], i \in [N]$, we define

$$\begin{aligned} S_T^{\mathbf{x}} \triangleq \sum_{t=2}^{T} \|\mathbf{x}_t - \mathbf{x}_{t-1}\|^2,\ S_{T,i}^{\mathbf{x}} \triangleq \sum_{t=2}^{T} \|\mathbf{x}_{t,i} - \mathbf{x}_{t-1,i}\|^2, \\ S_T^{\text{TOP}} \triangleq \sum_{t=2}^{T} \|\boldsymbol{q}_t^{\text{TOP}} - \boldsymbol{q}_{t-1}^{\text{TOP}}\|_1^2, \quad \text{and} \quad S_{T,j}^{\text{MID}} \triangleq \sum_{t=2}^{T} \|\boldsymbol{q}_{t,j}^{\text{MID}} - \boldsymbol{q}_{t-1,j}^{\text{MID}}\|_1^2. \end{aligned} \tag{A.1}$$

A.1 Proof of Lemma 2

In this part, we analyze the negative stability terms in the MsMwC algorithm (Chen et al., 2021). For self-containedness, we restate its update rule in the following general form:

$$\boldsymbol{p}_t = \operatorname*{arg\,min}_{\boldsymbol{p} \in \Delta_d} \left\{ \langle \boldsymbol{m}_t, \boldsymbol{p} \rangle + \mathcal{D}_{\psi_t}(\boldsymbol{p}, \widehat{\boldsymbol{p}}_t) \right\}, \quad \widehat{\boldsymbol{p}}_{t+1} = \operatorname*{arg\,min}_{\boldsymbol{p} \in \Delta_d} \left\{ \langle \boldsymbol{\ell}_t + \boldsymbol{b}_t, \boldsymbol{p} \rangle + \mathcal{D}_{\psi_t}(\boldsymbol{p}, \widehat{\boldsymbol{p}}_t) \right\},$$

where Δ_d denotes a d-dimension simplex, $\psi_t(\boldsymbol{p}) = \sum_{i=1}^{d} \varepsilon_{t,i}^{-1} p_i \log p_i$ is the weighted negative entropy regularizer with time-coordinate-varying learning rate $\varepsilon_{t,i}$, and the bias term $a_{t,i} = 16\varepsilon_{t,i}(\ell_{t,i} - m_{t,i})^2$. Below, we give a detailed proof of Lemma 2, following a similar logic flow as Lemma 1 of Chen et al. (2021), while illustrating the negative stability terms. Moreover, for generality, we investigate a more general setting of an arbitrary comparator $\boldsymbol{u} \in \Delta_d$ and changing step sizes $\varepsilon_{t,i}$. This was done hoping that the negative stability term analysis would be comprehensive enough for readers interested solely in the MsMwC algorithm.

Proof [of Lemma 2] To begin with, the regret with correction can be analyzed as follows:

$$\begin{aligned} \sum_{t=1}^{T} \langle \boldsymbol{\ell}_t + \boldsymbol{b}_t, \boldsymbol{p}_t - \boldsymbol{u} \rangle &\le \sum_{t=1}^{T} \left(\mathcal{D}_{\psi_t}(\boldsymbol{u}, \widehat{\boldsymbol{p}}_t) - \mathcal{D}_{\psi_t}(\boldsymbol{u}, \widehat{\boldsymbol{p}}_{t+1})\right) + \sum_{t=1}^{T} \langle \boldsymbol{\ell}_t + \boldsymbol{b}_t - \boldsymbol{m}_t, \boldsymbol{p}_t - \widehat{\boldsymbol{p}}_{t+1} \rangle \\ &\quad - \sum_{t=1}^{T} \left(\mathcal{D}_{\psi_t}(\widehat{\boldsymbol{p}}_{t+1}, \boldsymbol{p}_t) + \mathcal{D}_{\psi_t}(\boldsymbol{p}_t, \widehat{\boldsymbol{p}}_t)\right) \\ &\le \underbrace{\sum_{t=1}^{T} \left(\mathcal{D}_{\psi_t}(\boldsymbol{u}, \widehat{\boldsymbol{p}}_t) - \mathcal{D}_{\psi_t}(\boldsymbol{u}, \widehat{\boldsymbol{p}}_{t+1})\right)}_{\text{TERM (A)}} + \underbrace{\sum_{t=1}^{T} \left(\langle \boldsymbol{\ell}_t + \boldsymbol{b}_t - \boldsymbol{m}_t, \boldsymbol{p}_t - \widehat{\boldsymbol{p}}_{t+1} \rangle - \frac{1}{2} \mathcal{D}_{\psi_t}(\widehat{\boldsymbol{p}}_{t+1}, \boldsymbol{p}_t) \right)}_{\text{TERM (B)}} \\ &\quad \underbrace{- \frac{1}{2} \sum_{t=1}^{T} \left(\mathcal{D}_{\psi_t}(\widehat{\boldsymbol{p}}_{t+1}, \boldsymbol{p}_t) + \mathcal{D}_{\psi_t}(\boldsymbol{p}_t, \widehat{\boldsymbol{p}}_t)\right)}_{\text{TERM (C)}}, \end{aligned}$$

where the first step follows the standard analysis of OOMD, e.g., Theorem 1 of Zhao et al. (2024). One difference of our analysis from the previous one lies in the second step, where previous work dropped the $\mathcal{D}_{\psi_t}(\boldsymbol{p}_t, \widehat{\boldsymbol{p}}_t)$ term while we keep it for negative terms.

To begin with, we require an upper bound of $\varepsilon_{t,i} \leq \frac{1}{32}$ for the step sizes. To give a lower bound for TERM (C), we notice that for any $\boldsymbol{a}, \boldsymbol{b} \in \Delta_d$,

$$\begin{aligned}\mathcal{D}_{\psi_t}(\boldsymbol{a}, \boldsymbol{b}) = &\sum_{i=1}^d \frac{1}{\varepsilon_{t,i}} \left(a_i \log \frac{a_i}{b_i} - a_i + b_i \right) = \sum_{i=1}^d \frac{b_i}{\varepsilon_{t,i}} \left(\frac{a_i}{b_i} \log \frac{a_i}{b_i} - \frac{a_i}{b_i} + 1 \right) \\ &\geq \min_{t,i} \frac{1}{\varepsilon_{t,i}} \sum_{i=1}^d \left(a_i \log \frac{a_i}{b_i} - a_i + b_i \right) \geq 32 \mathrm{KL}(\boldsymbol{a}, \boldsymbol{b}), \end{aligned} \tag{A.2}$$

where the first inequality is due to $x \log x - x + 1 \geq 0$ for all $x > 0$ and the last step is by $\varepsilon_{t,i} \leq \frac{1}{32}$. Thus, we have

$$\begin{aligned}\text{TERM (C)} &\geq 32 \sum_{t=1}^T (\mathrm{KL}(\widehat{\boldsymbol{p}}_{t+1}, \boldsymbol{p}_t) + \mathrm{KL}(\boldsymbol{p}_t, \widehat{\boldsymbol{p}}_t)) \geq \frac{32}{2 \log 2} \sum_{t=1}^T \left(\|\widehat{\boldsymbol{p}}_{t+1} - \boldsymbol{p}_t\|_1^2 + \|\boldsymbol{p}_t - \widehat{\boldsymbol{p}}_t\|_1^2 \right) \\ &\geq 16 \sum_{t=2}^T \left(\|\widehat{\boldsymbol{p}}_t - \boldsymbol{p}_{t-1}\|_1^2 + \|\boldsymbol{p}_t - \widehat{\boldsymbol{p}}_t\|_1^2 \right) \geq 8 \sum_{t=2}^T \|\boldsymbol{p}_t - \boldsymbol{p}_{t-1}\|_1^2, \end{aligned}$$

where the first step is from the above derivation, the second step is due to the Pinsker's inequality (Pinsker, 1964): $\mathrm{KL}(\boldsymbol{a}, \boldsymbol{b}) \geq \frac{1}{2 \log 2} \|\boldsymbol{a} - \boldsymbol{b}\|_1^2$ for any $\boldsymbol{a}, \boldsymbol{b} \in \Delta_d$.

For TERM (B), the proof is similar to the previous work, where only some constants are modified. For self-containedness, we give the analysis below. Treating $\widehat{\boldsymbol{p}}_{t+1}$ as a free variable and defining

$$\boldsymbol{p}^\star \in \arg\max_{\boldsymbol{p}} \langle \boldsymbol{\ell}_t + \boldsymbol{b}_t - \boldsymbol{m}_t, \boldsymbol{p}_t - \boldsymbol{p} \rangle - \frac{1}{2} \mathcal{D}_{\psi_t}(\boldsymbol{p}, \boldsymbol{p}_t),$$

by the optimality of $\boldsymbol{p}^\star$, we have

$$\boldsymbol{\ell}_t + \boldsymbol{b}_t - \boldsymbol{m}_t = \frac{1}{2} (\nabla \psi_t(\boldsymbol{p}_t) - \nabla \psi_t(\boldsymbol{p}^\star)).$$

Since $[\nabla \psi_t(\boldsymbol{p})]_i = \frac{1}{\varepsilon_{t,i}} (\log p_i + 1)$, it holds that

$$\ell_{t,i} - m_{t,i} + b_{t,i} = \frac{1}{2\varepsilon_{t,i}} \log \frac{p_{t,i}}{p_i^\star} \Leftrightarrow p_i^\star = p_{t,i} \exp(-2\varepsilon_{t,i}(\ell_{t,i} - m_{t,i} + b_{t,i})).$$

Therefore we have

$$\begin{aligned}&\langle \boldsymbol{\ell}_t + \boldsymbol{b}_t - \boldsymbol{m}_t, \boldsymbol{p}_t - \widehat{\boldsymbol{p}}_{t+1} \rangle - \frac{1}{2} \mathcal{D}_{\psi_t}(\widehat{\boldsymbol{p}}_{t+1}, \boldsymbol{p}_t) \leq \langle \boldsymbol{\ell}_t + \boldsymbol{b}_t - \boldsymbol{m}_t, \boldsymbol{p}_t - \boldsymbol{p}^\star \rangle - \frac{1}{2} \mathcal{D}_{\psi_t}(\boldsymbol{p}^\star, \boldsymbol{p}_t) \\ &= \frac{1}{2} \langle \nabla \psi_t(\boldsymbol{p}_t) - \nabla \psi_t(\boldsymbol{p}^\star), \boldsymbol{p}_t - \boldsymbol{p}^\star \rangle - \frac{1}{2} \mathcal{D}_{\psi_t}(\boldsymbol{p}^\star, \boldsymbol{p}_t) = \frac{1}{2} \mathcal{D}_{\psi_t}(\boldsymbol{p}_t, \boldsymbol{p}^\star) \qquad \text{(by definition)} \\ &= \frac{1}{2} \sum_{i=1}^d \frac{1}{\varepsilon_{t,i}} \left(p_{t,i} \log \frac{p_{t,i}}{p_i^\star} - p_{t,i} + p_i^\star \right) \\ &= \frac{1}{2} \sum_{i=1}^d \frac{p_{t,i}}{\varepsilon_{t,i}} (2\varepsilon_{t,i}(\ell_{t,i} - m_{t,i} + b_{t,i}) - 1 + \exp(-2\varepsilon_{t,i}(\ell_{t,i} - m_{t,i} + b_{t,i}))) \\ &\leq \frac{1}{2} \sum_{i=1}^d \frac{p_{t,i}}{\varepsilon_{t,i}} 4\varepsilon_{t,i}^2 (\ell_{t,i} - m_{t,i} + b_{t,i})^2 = 2 \sum_{i=1}^d \varepsilon_{t,i} p_{t,i} (\ell_{t,i} - m_{t,i} + b_{t,i})^2, \end{aligned}$$

where the first and second steps use the optimality of $\boldsymbol{p}^\star$, the last inequality uses $e^{-x} - 1 + x \leq x^2$ for all $x \geq -1$, requiring $|2\varepsilon_{t,i}(\ell_{t,i} - m_{t,i} + b_{t,i})| \leq 1$. It can be satisfied by $\varepsilon_{t,i} \leq 1/32$ and $|\ell_{t,i} - m_{t,i} + b_{t,i}| \leq 16$, where the latter requirement can be satisfied by setting $b_{t,i} = 16\varepsilon_{t,i}(\ell_{t,i} - m_{t,i})^2$:

$$|\ell_{t,i} - m_{t,i} + b_{t,i}| \leq 2 + 16 \cdot \frac{1}{32}(2\ell_{t,i}^2 + 2m_{t,i}^2) \leq 4 \leq 16.$$

As a result, we have

$$(\ell_{t,i} - m_{t,i} + b_{t,i})^2 = \Big(\ell_{t,i} - m_{t,i} + 16\varepsilon_{t,i}(\ell_{t,i} - m_{t,i})^2\Big)^2 \leq 4(\ell_{t,i} - m_{t,i})^2,$$

where the last step holds because $|\ell_{t,i}|, |m_{t,i}| \leq 1$ and $\varepsilon_{t,i} \leq 1/32$. Finally, it holds that

$$\textsc{term (b)} \leq 2\sum_{t=1}^T \sum_{i=1}^d \varepsilon_{t,i} p_{t,i}(\ell_{t,i} - m_{t,i} + b_{t,i})^2 \leq 8\sum_{t=1}^T \sum_{i=1}^d \varepsilon_{t,i} p_{t,i}(\ell_{t,i} - m_{t,i})^2.$$

As for $\textsc{term (a)}$, following the same argument as Lemma 1 of Chen et al. (2021), we have

$$\textsc{term (a)} \leq \sum_{i=1}^d \frac{1}{\varepsilon_{1,i}} f_{\mathrm{KL}}(u_i, \widehat{p}_{1,i}) + \sum_{t=2}^T \sum_{i=1}^d \left(\frac{1}{\varepsilon_{t,i}} - \frac{1}{\varepsilon_{t-1,i}}\right) f_{\mathrm{KL}}(u_i, \widehat{p}_{t,i}),$$

where $f_{\mathrm{KL}}(a,b) \triangleq a\log(a/b) - a + b$. Combining all three terms, we have

$$\begin{aligned}\sum_{t=1}^T \langle \boldsymbol{\ell}_t + \boldsymbol{b}_t, \boldsymbol{p}_t - \boldsymbol{u}\rangle \leq\ & \sum_{i=1}^d \frac{1}{\varepsilon_{1,i}} f_{\mathrm{KL}}(u_i, \widehat{p}_{1,i}) + \sum_{t=2}^T \sum_{i=1}^d \left(\frac{1}{\varepsilon_{t,i}} - \frac{1}{\varepsilon_{t-1,i}}\right) f_{\mathrm{KL}}(u_i, \widehat{p}_{t,i}) \\ & + 8\sum_{t=1}^T \sum_{i=1}^d \varepsilon_{t,i} p_{t,i}(\ell_{t,i} - m_{t,i})^2 - 4\sum_{t=2}^T \|\boldsymbol{p}_t - \boldsymbol{p}_{t-1}\|_1^2.\end{aligned}$$

Moving the correction term $\sum_{t=1}^T \langle \boldsymbol{b}_t, \boldsymbol{p}_t - \boldsymbol{u}\rangle$ to the right-hand side gives:

$$\begin{aligned}\sum_{t=1}^T \langle \boldsymbol{\ell}_t, \boldsymbol{p}_t - \boldsymbol{u}\rangle \leq\ & \sum_{i=1}^d \frac{1}{\varepsilon_{1,i}} f_{\mathrm{KL}}(u_i, \widehat{p}_{1,i}) + \sum_{t=2}^T \sum_{i=1}^d \left(\frac{1}{\varepsilon_{t,i}} - \frac{1}{\varepsilon_{t-1,i}}\right) f_{\mathrm{KL}}(u_i, \widehat{p}_{t,i}) \\ & - 8\sum_{t=1}^T \sum_{i=1}^d \varepsilon_{t,i} p_{t,i}(\ell_{t,i} - m_{t,i})^2 + 16\sum_{t=1}^T \sum_{i=1}^d \varepsilon_{t,i} u_i(\ell_{t,i} - m_{t,i})^2 - 4\sum_{t=2}^T \|\boldsymbol{p}_t - \boldsymbol{p}_{t-1}\|_1^2.\end{aligned}$$

Finally, choosing $\boldsymbol{u} = \boldsymbol{e}_{i^\star}$ and $\varepsilon_{t,i} = \varepsilon_i$ for all $t \in [T]$ finishes the proof. ■

A.2 Proof of Lemma 3

Proof For simplicity, we introduce the notation $\mathbf{g}_t = \nabla f_t(\mathbf{x}_t)$ to denote the gradient at each round. Below we consider the three function families respectively.

For *exp-concave* and *strongly convex* functions, we have

$$\begin{aligned}\sum_{t=1}^{T}(r_{t,i^\star}-m_{t,i^\star})^2 &= \sum_{t=1}^{T}(\langle \mathbf{g}_t, \mathbf{x}_t-\mathbf{x}_{t,i^\star}\rangle - \langle \mathbf{g}_{t-1}, \mathbf{x}_{t-1}-\mathbf{x}_{t-1,i^\star}\rangle)^2\\ &\le 2\sum_{t=1}^{T}\langle \mathbf{g}_t, \mathbf{x}_t-\mathbf{x}_{t,i^\star}\rangle^2 + 2\sum_{t=1}^{T}\langle \mathbf{g}_{t-1}, \mathbf{x}_{t-1}-\mathbf{x}_{t-1,i^\star}\rangle^2\\ &\le 4\sum_{t=1}^{T}\langle \mathbf{g}_t, \mathbf{x}_t-\mathbf{x}_{t,i^\star}\rangle^2 + \mathcal{O}(1).\end{aligned}$$

For *strongly convex* functions, using the boundedness of gradients, we further obtain

$$\sum_{t=1}^{T}(r_{t,i^\star}-m_{t,i^\star})^2 = 4\sum_{t=1}^{T}\langle \mathbf{g}_t, \mathbf{x}_t-\mathbf{x}_{t,i^\star}\rangle^2 + \mathcal{O}(1) \le 4G^2\sum_{t=1}^{T}\|\mathbf{x}_t-\mathbf{x}_{t,i^\star}\|^2 + \mathcal{O}(1).$$

For *convex* function, it holds that

$$\begin{aligned}&\sum_{t=1}^{T}(r_{t,i^\star}-m_{t,i^\star})^2 = \sum_{t=1}^{T}\Big(\langle \mathbf{g}_t, \mathbf{x}_t-\mathbf{x}_{t,i^\star}\rangle - \langle \mathbf{g}_{t-1}, \mathbf{x}_{t-1}-\mathbf{x}_{t-1,i^\star}\rangle\Big)^2\\ &\le 2\sum_{t=1}^{T}\langle \mathbf{g}_t-\mathbf{g}_{t-1}, \mathbf{x}_t-\mathbf{x}_{t,i^\star}\rangle^2 + 2\sum_{t=1}^{T}\langle \mathbf{g}_{t-1}, \mathbf{x}_t-\mathbf{x}_{t-1}+\mathbf{x}_{t,i^\star}-\mathbf{x}_{t-1,i^\star}\rangle^2\\ &\le 2D^2\sum_{t=1}^{T}\|\mathbf{g}_t-\mathbf{g}_{t-1}\|^2 + 2G^2\sum_{t=1}^{T}\|\mathbf{x}_t-\mathbf{x}_{t-1}+\mathbf{x}_{t,i^\star}-\mathbf{x}_{t-1,i^\star}\|^2 \qquad \text{(A.3)}\\ &\le 4D^2V_T + 4(D^2L^2+G^2)S_T^{\mathbf{x}} + 4G^2S_{T,i^\star}^{\mathbf{x}},\end{aligned}$$

where the fourth step is by Assumption 1 and Assumption 2 and the last step is due to the definition of the gradient variation, finishing the proof. ■

A.3 Proof of Lemma 4

Proof By (A.2), the regret of MoM-Top can be bounded as

$$\sum_{t=1}^{T}\langle \boldsymbol{\ell}_t^{\text{TOP}}, \boldsymbol{q}_t^{\text{TOP}}-\boldsymbol{e}_{j^\star}\rangle \le \left(\frac{1}{\varepsilon_{j^\star}^{\text{TOP}}}\log\frac{1}{\widehat{q}_{1,j^\star}^{\text{TOP}}} + \sum_{j=1}^{M}\frac{\widehat{q}_{1,j}^{\text{TOP}}}{\varepsilon_j^{\text{TOP}}}\right) + 16\varepsilon_{j^\star}^{\text{TOP}}\sum_{t=1}^{T}(\ell_{t,j^\star}^{\text{TOP}}-m_{t,j^\star}^{\text{TOP}})^2 - \min_{j\in[M]}\frac{1}{4\varepsilon_j^{\text{TOP}}}S_T^{\text{TOP}},$$

where the first step comes from $f_{\text{KL}}(a,b) = a\log(a/b)-a+b \le a\log(a/b)+b$ for $a,b>0$. The first term above can be further bounded as

$$\frac{1}{\varepsilon_{j^\star}^{\text{TOP}}}\log\frac{1}{\widehat{q}_{1,j^\star}^{\text{TOP}}} + \sum_{j=1}^{M}\frac{\widehat{q}_{1,j}^{\text{TOP}}}{\varepsilon_j^{\text{TOP}}} = \frac{1}{\varepsilon_{j^\star}^{\text{TOP}}}\log\frac{\sum_{j=1}^{M}(\varepsilon_j^{\text{TOP}})^2}{(\varepsilon_{j^\star}^{\text{TOP}})^2} + \frac{\sum_{j=1}^{M}\varepsilon_j^{\text{TOP}}}{\sum_{j=1}^{M}(\varepsilon_j^{\text{TOP}})^2} \le \frac{1}{\varepsilon_{j^\star}^{\text{TOP}}}\log\frac{1}{3C_0^2(\varepsilon_{j^\star}^{\text{TOP}})^2} + 4C_0,$$

where the first step is due to the initialization of $\widehat{q}_{1,j}^{\text{TOP}} = (\varepsilon_j^{\text{TOP}})^2/\sum_{j=1}^{M}(\varepsilon_j^{\text{TOP}})^2$. Plugging in the setting of $\varepsilon_j^{\text{TOP}} = 1/(C_0\cdot 2^j)$, the second step holds since

$$\frac{1}{4C_0^2} = \sum_{j=1}^{M}(\varepsilon_j^{\text{TOP}})^2 \le \sum_{j=1}^{M}\frac{1}{C_0^2\cdot 4^j} \le \frac{1}{3C_0^2}, \quad \sum_{j=1}^{M}\varepsilon_j^{\text{TOP}} = \sum_{j=1}^{M}\frac{1}{C_0\cdot 2^j} \le \frac{1}{C_0}.$$

Since $1/\varepsilon_j^{\text{TOP}} = C_0 \cdot 2^j \geq 2C_0$, the regret of MoM-Top can be bounded by

$$\sum_{t=1}^T \langle \boldsymbol{\ell}_t^{\text{TOP}}, \boldsymbol{q}_t^{\text{TOP}} - \boldsymbol{e}_{j^\star} \rangle \leq \frac{1}{\varepsilon_{j^\star}^{\text{TOP}}} \log \frac{1}{3C_0^2(\varepsilon_{j^\star}^{\text{TOP}})^2} + 16\varepsilon_{j^\star}^{\text{TOP}} \sum_{t=1}^T (\ell_{t,j^\star}^{\text{TOP}} - m_{t,j^\star}^{\text{TOP}})^2 - \frac{C_0}{2} S_T^{\text{TOP}} + \mathcal{O}(1).$$

Next, using Lemma 2 again, the regret of the $j^\star$-th MoM-Mid, whose step size is $\varepsilon_{t,j,i}^{\text{MID}} = 2\varepsilon_j^{\text{TOP}}$ for all $t \in [T]$ and $i \in [N]$, can be bounded as

$$\begin{aligned}\sum_{t=1}^T \langle \boldsymbol{\ell}_{t,j^\star}^{\text{MID}}, \boldsymbol{q}_{t,j^\star}^{\text{MID}} - \boldsymbol{e}_{i^\star} \rangle \leq{}& \frac{\log N}{2\varepsilon_{j^\star}^{\text{TOP}}} + 32\varepsilon_{j^\star}^{\text{TOP}} \sum_{t=1}^T (\ell_{t,j^\star,i^\star}^{\text{MID}} - m_{t,j^\star,i^\star}^{\text{MID}})^2 - \frac{C_0}{4} S_{T,j^\star}^{\text{MID}} \\ &- 16\varepsilon_{j^\star}^{\text{TOP}} \sum_{t=1}^T \sum_{i=1}^N q_{t,j^\star,i}^{\text{MID}} (\ell_{t,j,i}^{\text{MID}} - m_{t,j,i}^{\text{MID}})^2,\end{aligned}$$

where the first step is due to the initialization of $\widehat{p}_{1,j,i}^{\text{MID}} = 1/N$. Based on the observation of

$$(\ell_{t,j}^{\text{TOP}} - m_{t,j}^{\text{TOP}})^2 = \langle \boldsymbol{\ell}_{t,j}^{\text{MID}} - \boldsymbol{m}_{t,j}^{\text{MID}}, \boldsymbol{q}_{t,j}^{\text{MID}} \rangle^2 \leq \sum_{i=1}^N q_{t,j^\star,i}^{\text{MID}} (\ell_{t,j,i}^{\text{MID}} - m_{t,j,i}^{\text{MID}})^2,$$

where the last step uses the Cauchy-Schwarz inequality, combining the regret of MoM-Top and the $j^\star$-th MoM-Mid finishes the proof. ■

A.4 Proof of Theorem 1

Proof The proof proceeds in three steps: we first decompose the total regret into meta and base regret, then analyze the meta regret and base regret separately, and finally combines them to achieve the final regret guarantees.

Regret Decomposition. For simplicity, we let $\mathbf{x}^\star = \arg\min_{\mathcal{X}} f_t(\mathbf{x})$ and $\mathbf{g}_t = \nabla f_t(\mathbf{x}_t)$. For λ-*strongly convex* functions, we decompose the regret as

$$\text{Reg}_T \leq \underbrace{\sum_{t=1}^T \langle \mathbf{g}_t, \mathbf{x}_t - \mathbf{x}_{t,i^\star} \rangle - \frac{\lambda}{2} \sum_{t=1}^T \|\mathbf{x}_t - \mathbf{x}_{t,i^\star}\|^2}_{\text{Meta-Reg}} + \underbrace{\sum_{t=1}^T f_t(\mathbf{x}_{t,i^\star}) - \sum_{t=1}^T f_t(\mathbf{x}^\star)}_{\text{Base-Reg}},$$

where $\lambda_{i^\star} \leq \lambda \leq 2\lambda_{i^\star}$. For α-*exp-concave* functions, we decompose the regret as

$$\text{Reg}_T \leq \underbrace{\sum_{t=1}^T \langle \mathbf{g}_t, \mathbf{x}_t - \mathbf{x}_{t,i} \rangle - \frac{\alpha}{2} \sum_{t=1}^T \langle \mathbf{g}_t, \mathbf{x}_t - \mathbf{x}_{t,i^\star} \rangle^2}_{\text{Meta-Reg}} + \underbrace{\sum_{t=1}^T f_t(\mathbf{x}_{t,i^\star}) - \sum_{t=1}^T f_t(\mathbf{x}^\star)}_{\text{Base-Reg}},$$

where $\alpha_{i^\star} \leq \alpha \leq 2\alpha_{i^\star}$. For *convex* functions, we decompose the regret as

$$\text{Reg}_T = \underbrace{\sum_{t=1}^T f_t(\mathbf{x}_t) - \sum_{t=1}^T f_t(\mathbf{x}_{t,i^\star})}_{\text{Meta-Reg}} + \underbrace{\sum_{t=1}^T f_t(\mathbf{x}_{t,i^\star}) - \sum_{t=1}^T f_t(\mathbf{x}^\star)}_{\text{Base-Reg}}.$$

Meta Regret Analysis. Recall that the normalization factor $Z = \max\{GD+\gamma^{\text{MID}} D^2, 1+\gamma^{\text{MID}} D^2 + 2\gamma^{\text{TOP}}\}$. We focus on the linearized term $\sum_{t=1}^T \langle \mathbf{g}_t, \mathbf{x}_t - \mathbf{x}_{t,i^\star}\rangle$ and let $V_\star = \sum_{t=2}^T (\ell_{t,j^\star,i^\star}^{\text{MID}} - m_{t,j^\star,i^\star}^{\text{MID}})^2$. Specifically,

$$
\begin{aligned}
&\sum_{t=1}^T \langle \mathbf{g}_t, \mathbf{x}_t - \mathbf{x}_{t,i^\star}\rangle = Z\sum_{t=1}^T \langle \boldsymbol{\ell}_t, \boldsymbol{p}_t - \boldsymbol{e}_{i^\star}\rangle = Z\sum_{t=1}^T \langle \boldsymbol{\ell}_t, \boldsymbol{p}_t - \boldsymbol{q}_{t,j^\star}^{\text{MID}}\rangle + Z\sum_{t=1}^T \langle \boldsymbol{\ell}_t, \boldsymbol{q}_{t,j^\star}^{\text{MID}} - \boldsymbol{e}_{i^\star}\rangle \\
&= Z\sum_{t=1}^T \langle \boldsymbol{\ell}_t^{\text{TOP}}, \boldsymbol{q}_t^{\text{TOP}} - \boldsymbol{e}_{j^\star}\rangle + Z\sum_{t=1}^T \langle \boldsymbol{\ell}_t^{\text{MID}}, \boldsymbol{q}_{t,j^\star}^{\text{MID}} - \boldsymbol{e}_{i^\star}\rangle - \gamma^{\text{TOP}} \sum_{t=2}^T\sum_{j=1}^M q_{t,j}^{\text{TOP}} \|\boldsymbol{q}_{t,j}^{\text{MID}} - \boldsymbol{q}_{t-1,j}^{\text{MID}}\|_1^2 \\
&\qquad + \gamma^{\text{TOP}} S_{T,j^\star}^{\text{MID}} - \gamma^{\text{MID}} \sum_{t=2}^T\sum_{j=1}^M q_{t,j}^{\text{TOP}} \sum_{i=1}^N q_{t,j,i}^{\text{MID}} \|\mathbf{x}_{t,i} - \mathbf{x}_{t-1,i}\|^2 + \gamma^{\text{MID}} S_{T,i^\star}^{\mathbf{x}} \\
&\qquad + \gamma^{\text{MID}} \sum_{t=2}^T\sum_{i=1}^N q_{t,j^\star,i}^{\text{MID}} \|\mathbf{x}_{t,i} - \mathbf{x}_{t-1,i}\|^2 - \gamma^{\text{MID}} \sum_{t=2}^T\sum_{i=1}^N q_{t,j^\star,i}^{\text{MID}} \|\mathbf{x}_{t,i} - \mathbf{x}_{t-1,i}\|^2 \\
&\le \frac{Z}{\varepsilon_{j^\star}^{\text{TOP}}} \log \frac{N}{3C_0^2(\varepsilon_{j^\star}^{\text{TOP}})^2} + 32Z\varepsilon_{j^\star}^{\text{TOP}} V_\star - \frac{C_0}{2} S_T^{\text{TOP}} + \left(\gamma^{\text{TOP}} - \frac{C_0}{4}\right) S_{T,j^\star}^{\text{MID}} + \gamma^{\text{MID}} S_{T,i^\star}^{\mathbf{x}} \\
&\qquad - \gamma^{\text{MID}} \sum_{t=2}^T\sum_{j=1}^M q_{t,j}^{\text{TOP}} \sum_{i=1}^N q_{t,j,i}^{\text{MID}} \|\mathbf{x}_{t,i} - \mathbf{x}_{t-1,i}\|^2 - \gamma^{\text{TOP}} \sum_{t=2}^T\sum_{j=1}^M q_{t,j}^{\text{TOP}} \|\boldsymbol{q}_{t,j}^{\text{MID}} - \boldsymbol{q}_{t-1,j}^{\text{MID}}\|_1^2 \\
&\le \frac{Z}{\varepsilon_{j^\star}^{\text{TOP}}} \log \frac{N}{3C_0^2(\varepsilon_{j^\star}^{\text{TOP}})^2} + 32Z\varepsilon_{j^\star}^{\text{TOP}} V_\star - \frac{C_0}{2} S_T^{\text{TOP}} + \gamma^{\text{MID}} S_{T,i^\star}^{\mathbf{x}} \qquad \text{(requiring } C_0 \ge 4\gamma^{\text{TOP}}\text{)} \\
&\qquad - \gamma^{\text{MID}} \sum_{t=2}^T\sum_{j=1}^M q_{t,j}^{\text{TOP}} \sum_{i=1}^N q_{t,j,i}^{\text{MID}} \|\mathbf{x}_{t,i} - \mathbf{x}_{t-1,i}\|^2 - \gamma^{\text{TOP}} \sum_{t=2}^T\sum_{j=1}^M q_{t,j}^{\text{TOP}} \|\boldsymbol{q}_{t,j}^{\text{MID}} - \boldsymbol{q}_{t-1,j}^{\text{MID}}\|_1^2, \qquad \text{(A.4)}
\end{aligned}
$$

where the first step is due to $\mathbf{x}_t = \sum_{i=1}^N p_{t,i}\mathbf{x}_{t,i}$ and defines $\ell_{t,i} \triangleq \frac{1}{Z}\langle \nabla f_t(\mathbf{x}_t), \mathbf{x}_{t,i}\rangle$. The third step is due to the definition of $\boldsymbol{\ell}_t^{\text{TOP}}$ and $\boldsymbol{\ell}_t^{\text{MID}}$ as defined in Eq. (3.16) and Eq. (3.17). The fourth step uses the analysis of MoM as show in in Lemma 4.

For λ-*strongly convex* functions, applying Eq. (A.4) and omitting the stability and curvature-induced negative terms, we bound the meta regret by

$$
\begin{aligned}
&\text{Meta-Reg} \le \sum_{t=1}^T \langle \mathbf{g}_t, \mathbf{x}_t - \mathbf{x}_{t,i^\star}\rangle - \frac{\lambda}{2}\sum_{t=1}^T \|\mathbf{x}_t - \mathbf{x}_{t,i^\star}\|^2 \\
&\le \frac{Z}{\varepsilon_{j^\star}^{\text{TOP}}} \log \frac{N}{3C_0^2(\varepsilon_{j^\star}^{\text{TOP}})^2} + 32Z\varepsilon_{j^\star}^{\text{TOP}} V_\star + \gamma^{\text{MID}} S_{T,i^\star}^{\mathbf{x}} - \frac{\lambda}{2}\sum_{t=1}^T \|\mathbf{x}_t - \mathbf{x}_{t,i^\star}\|^2 \\
&\le \frac{Z}{\varepsilon_{j^\star}^{\text{TOP}}} \log \frac{N}{3C_0^2(\varepsilon_{j^\star}^{\text{TOP}})^2} + \left(\frac{128G^2}{Z}\varepsilon_{j^\star}^{\text{TOP}} - \frac{\lambda}{2}\right)\sum_{t=1}^T \|\mathbf{x}_t - \mathbf{x}_{t,i^\star}\|^2 + \gamma^{\text{MID}} S_{T,i^\star}^{\mathbf{x}} \\
&\le 2ZC_0 \log \frac{4N}{3} + \frac{512ZG^2}{\lambda} \log \frac{2^{20}G^2N}{3C_0^2\lambda^2} + \gamma^{\text{MID}} S_{T,i^\star}^{\mathbf{x}}, \qquad \text{(A.5)}
\end{aligned}
$$

where the third step leverages the property of our universal optimism design as given in Lemma 3 and the last step again follows from Lemma 13 and requires $\varepsilon_{j^\star}^{\text{TOP}} \le \varepsilon_\star^{\text{TOP}} \triangleq \frac{\lambda Z}{256G^2}$.

For α-*exp-concave* functions, applying Eq. (A.4) and omitting the stability and curvature-induced negative terms, we bound the meta regret by

$$
\begin{aligned}
\textsc{Meta-Reg} &\le \sum_{t=1}^T \langle \mathbf{g}_t, \mathbf{x}_t - \mathbf{x}_{t,i^\star}\rangle - \frac{\alpha}{2}\sum_{t=1}^T \langle \mathbf{g}_t, \mathbf{x}_t - \mathbf{x}_{t,i^\star}\rangle^2 \\
&\le \frac{Z}{\varepsilon_{j^\star}^{\text{TOP}}} \log \frac{N}{3C_0^2(\varepsilon_{j^\star}^{\text{TOP}})^2} + 32Z\varepsilon_{j^\star}^{\text{TOP}} V_\star + \gamma^{\text{MID}} S_{T,i^\star}^{\mathbf{x}} - \frac{\alpha}{2}\sum_{t=1}^T \langle \mathbf{g}_t, \mathbf{x}_t - \mathbf{x}_{t,i^\star}\rangle^2 \\
&\le \frac{Z}{\varepsilon_{j^\star}^{\text{TOP}}} \log \frac{N}{3C_0^2(\varepsilon_{j^\star}^{\text{TOP}})^2} + \left(\frac{128\varepsilon_{j^\star}^{\text{TOP}}}{Z} - \frac{\alpha}{2}\right)\sum_{t=1}^T \langle \mathbf{g}_t, \mathbf{x}_t - \mathbf{x}_{t,i^\star}\rangle^2 + \gamma^{\text{MID}} S_{T,i^\star}^{\mathbf{x}} \\
&\le 2ZC_0 \log\frac{4N}{3} + \frac{512Z}{\alpha}\log\frac{2^{20}N}{3C_0^2\alpha^2} + \gamma^{\text{MID}} S_{T,i^\star}^{\mathbf{x}}, \qquad \text{(A.6)}
\end{aligned}
$$

where the third step leverages the property of our universal optimism design as given in Lemma 3 and the last step follows from Lemma 13 and requires $\varepsilon_{j^\star}^{\text{TOP}} \le \varepsilon_\star^{\text{TOP}} \triangleq \frac{\alpha Z}{256}$.

For *convex* functions, applying Eq. (A.4) while *retaining* the crucial stability and curvature-induced negative terms, the meta regret can be bounded by

$$
\begin{aligned}
\textsc{Meta-Reg} &\le \sum_{t=1}^T \langle \mathbf{g}_t, \mathbf{x}_t - \mathbf{x}_{t,i^\star}\rangle \\
&\overset{\text{(A.4)}}{\le} \frac{Z}{\varepsilon_{j^\star}^{\text{TOP}}} \log \frac{N}{3C_0^2(\varepsilon_{j^\star}^{\text{TOP}})^2} + 32Z\varepsilon_{j^\star}^{\text{TOP}} V_\star + \gamma^{\text{MID}} S_{T,i^\star}^{\mathbf{x}} - \frac{C_0}{2} S_T^{\text{TOP}} \\
&\quad - \gamma^{\text{MID}} \sum_{t=2}^T\sum_{j=1}^M q_{t,j}^{\text{TOP}} \sum_{i=1}^N q_{t,j,i}^{\text{MID}} \|\mathbf{x}_{t,i} - \mathbf{x}_{t-1,i}\|^2 - \gamma^{\text{TOP}} \sum_{t=2}^T\sum_{j=1}^M q_{t,j}^{\text{TOP}} \|\boldsymbol{q}_{t,j}^{\text{MID}} - \boldsymbol{q}_{t-1,j}^{\text{MID}}\|_1^2 \qquad \text{(A.7)} \\
&\le \frac{Z}{\varepsilon_{j^\star}^{\text{TOP}}} \log \frac{N}{3C_0^2(\varepsilon_{j^\star}^{\text{TOP}})^2} + \frac{128D^2\varepsilon_{j^\star}^{\text{TOP}}}{Z} V_T + \left(\frac{64G^2}{Z} + \gamma^{\text{MID}}\right) S_{T,i^\star}^{\mathbf{x}} + \frac{64(D^2L^2+G^2)}{Z} S_T^{\mathbf{x}} \\
&\quad - \frac{C_0}{2} S_T^{\text{TOP}} - \gamma^{\text{MID}} \sum_{t=2}^T\sum_{j=1}^M q_{t,j}^{\text{TOP}} \sum_{i=1}^N q_{t,j,i}^{\text{MID}} \|\mathbf{x}_{t,i} - \mathbf{x}_{t-1,i}\|^2 - \gamma^{\text{TOP}} \sum_{t=2}^T\sum_{j=1}^M q_{t,j}^{\text{TOP}} \|\boldsymbol{q}_{t,j}^{\text{MID}} - \boldsymbol{q}_{t-1,j}^{\text{MID}}\|_1^2 \\
&\le 2ZC_0 \log\frac{4N}{3} + 32D\sqrt{2V_T \log(512ND^2V_T/Z^2)} + \left(\frac{64G^2}{Z} + \gamma^{\text{MID}}\right) S_{T,i^\star}^{\mathbf{x}} + \frac{64(D^2L^2+G^2)}{Z} S_T^{\mathbf{x}} \\
&\quad - \frac{C_0}{2} S_T^{\text{TOP}} - \gamma^{\text{MID}} \sum_{t=2}^T\sum_{j=1}^M q_{t,j}^{\text{TOP}} \sum_{i=1}^N q_{t,j,i}^{\text{MID}} \|\mathbf{x}_{t,i} - \mathbf{x}_{t-1,i}\|^2 - \gamma^{\text{TOP}} \sum_{t=2}^T\sum_{j=1}^M q_{t,j}^{\text{TOP}} \|\boldsymbol{q}_{t,j}^{\text{MID}} - \boldsymbol{q}_{t-1,j}^{\text{MID}}\|_1^2 \\
&\qquad \text{(A.8)} \\
&\le 2ZC_0 \log\frac{4N}{3} + 32D\sqrt{2V_T \log(512ND^2V_T/Z^2)} + \left(\frac{64G^2}{Z} + \gamma^{\text{MID}}\right) S_{T,i^\star}^{\mathbf{x}} \\
&\quad + \left(\frac{C_1}{Z} - \gamma^{\text{MID}}\right) \sum_{t=2}^T\sum_{j=1}^M q_{t,j}^{\text{TOP}} \sum_{i=1}^N q_{t,j,i}^{\text{MID}} \|\mathbf{x}_{t,i} - \mathbf{x}_{t-1,i}\|^2 \\
&\quad + \left(\frac{2D^2C_1}{Z} - \frac{C_0}{2}\right) S_T^{\text{TOP}} + \left(\frac{2D^2C_1}{Z} - \gamma^{\text{TOP}}\right) \sum_{t=2}^T\sum_{j=1}^M q_{t,j}^{\text{TOP}} \|\boldsymbol{q}_{t,j}^{\text{MID}} - \boldsymbol{q}_{t-1,j}^{\text{MID}}\|_1^2 \\
&\qquad \text{(denoting } C_1 = 128(D^2L^2+G^2)\text{)}
\end{aligned}
$$

$$\le 2ZC_0 \log \frac{4N}{3} + 32D\sqrt{2V_T \log\left(512ND^2V_T/Z^2\right)} + \left(\frac{64G^2}{Z} + \gamma^{\text{MID}}\right) S^{\mathbf{x}}_{T,i^\star},$$
$$\text{(requiring } \gamma^{\text{TOP}} \ge 2D^2C_1/Z, \gamma^{\text{MID}} \ge C_1/Z, \text{ and } C_0 \ge 4D^2C_1/Z)$$

where the third step is due to Lemma 3, and $\varepsilon^{\text{TOP}}_{j^\star} \le 1/2$ under the requirement $C_0 \ge 1$. The fourth step is by Lemma 12 and requiring $C_0 \ge 8D$ and the fifth step is by Lemma 5.

Base Regret Analysis. For λ-*strongly convex* functions, the $i^\star$-th base learner guarantees the following:

$$\text{Base-Reg} \le \frac{16G^2}{\lambda_{i^\star}} \log\left(1 + 2\lambda_{i^\star}V_T + 2\lambda_{i^\star}L^2S^{\mathbf{x}}_{T,i^\star}\right) + \frac{1}{4}\kappa D^2 - \frac{1}{8}\kappa S^{\mathbf{x}}_{T,i^\star} + \mathcal{O}(1)$$
$$\le \frac{32G^2}{\lambda} \log\left(1 + 2\lambda V_T\right) + \left(32L^2G^2 - \frac{1}{8}\kappa\right) S^{\mathbf{x}}_{T,i^\star} + \frac{1}{4}\kappa D^2 + \mathcal{O}(1),$$

where the first step is due to Lemma 19 and $\sum_{t=2}^T \|\nabla f_t(\mathbf{x}_{t,i^\star}) - \nabla f_{t-1}(\mathbf{x}_{t-1,i^\star})\|^2 \le 2V_T + 2L^2S^{\mathbf{x}}_{T,i^\star}$ and the last step follows from $\log(1+x) \le x$ for $x \ge 0$ and $\lambda_{i^\star} \le \lambda \le 2\lambda_{i^\star}$.

For α-*exp-concave* functions, the $i^\star$-th base learner guarantees the following:

$$\text{Base-Reg} \le \frac{16d}{\alpha_{i^\star}} \log\left(1 + \frac{\alpha_{i^\star}}{4\kappa d}V_T + \frac{\alpha_{i^\star}L^2}{4\kappa d}S^{\mathbf{x}}_{T,i^\star}\right) + \frac{1}{2}\kappa D^2 - \frac{1}{4}\kappa S^{\mathbf{x}}_{T,i^\star} + \mathcal{O}(1)$$
$$\le \frac{32d}{\alpha} \log\left(1 + \frac{\alpha}{4\kappa d}V_T\right) + \left(\frac{4L^2}{\kappa} - \frac{1}{4}\kappa\right) S^{\mathbf{x}}_{T,i^\star} + \frac{1}{2}\kappa D^2 + \mathcal{O}(1),$$

where the first step is due to Lemma 20 and $\sum_{t=2}^T \|\nabla f_t(\mathbf{x}_{t,i^\star}) - \nabla f_{t-1}(\mathbf{x}_{t-1,i^\star})\|^2 \le 2V_T + 2L^2S^{\mathbf{x}}_{T,i^\star}$ and the last step follows from $\log(1+x) \le x$ for $x \ge 0$ and $\alpha_{i^\star} \le \alpha \le 2\alpha_{i^\star}$.

For *convex* functions, by Lemma 21, the $i^\star$-th base learner guarantees the following:

$$\text{Base-Reg} \le 5D\sqrt{1 + 2V_T + 2L^2S^{\mathbf{x}}_{T,i^\star}} + \kappa D^2 - \frac{1}{4}\kappa S^{\mathbf{x}}_{T,i^\star} + \mathcal{O}(1)$$
$$\le 5D\sqrt{1 + 2V_T} + \kappa D^2 + \left(10DL^2 - \frac{1}{4}\kappa\right) S^{\mathbf{x}}_{T,i^\star} + \mathcal{O}(1),$$

where the first step is due to $\sum_{t=2}^T \|\nabla f_t(\mathbf{x}_{t,i^\star}) - \nabla f_{t-1}(\mathbf{x}_{t-1,i^\star})\|^2 \le 2V_T + 2L^2S^{\mathbf{x}}_{T,i^\star}$.

Overall Regret Analysis. For λ-*strongly convex* functions, the overall regret can be bounded by

$$\text{Reg}_T \le 2ZC_0 \log \frac{4N}{3} + \frac{512ZG^2}{\lambda} \log \frac{2^{20}G^2N}{3C_0^2\lambda^2} + \frac{32G^2}{\lambda} \log\left(1 + 2\lambda V_T\right)$$
$$+ \left(\gamma^{\text{MID}} + 32L^2G^2 - \frac{\kappa}{4}\right) S^{\mathbf{x}}_{T,i^\star} + \frac{\kappa D^2}{4} + \mathcal{O}(1) \le \mathcal{O}\left(\frac{1}{\lambda}\log V_T\right),$$

where the last step requires $\kappa \ge 4\gamma^{\text{MID}} + 128L^2G^2$.

For α-*exp-concave* functions, the overall regret can be bounded by

$$\text{Reg}_T \le 2ZC_0 \log \frac{4N}{3} + \frac{512Z}{\alpha} \log \frac{2^{20}N}{3C_0^2\alpha^2} + \frac{32d}{\alpha} \log\left(1 + \frac{\alpha}{4\kappa d}V_T\right)$$
$$+ \left(\gamma^{\text{MID}} + \frac{4L^2}{\kappa} - \frac{1}{4}\kappa\right) S^{\mathbf{x}}_{T,i^\star} + \frac{1}{2}\kappa D^2 + \mathcal{O}(1) \le \mathcal{O}\left(\frac{d}{\alpha}\log V_T\right),$$

where the last step requires $\kappa \geq 4\gamma^{\text{MID}} + 8L^2$.

For *convex* functions, the overall regret can be bounded by

$$\begin{aligned}\textsc{Reg}_T \leq\ & 2ZC_0 \log \frac{4N}{3} + 32D\sqrt{2V_T \log\left(512ND^2V_T/Z^2\right)} + 5D\sqrt{1+2V_T} \\ & + \left(\frac{64G^2}{Z} + \gamma^{\text{MID}} + 10DL^2 - \frac{1}{4}\kappa\right) S^{\mathbf{x}}_{T,i^\star} + \kappa D^2 + \mathcal{O}(1) \leq \mathcal{O}\left(\sqrt{V_T \log V_T}\right),\end{aligned}$$

where the last step requires $\kappa \geq 4\gamma^{\text{MID}} + 40DL^2 + 256G^2/Z$.

At last, we determine the specific values of C_0, γ^{TOP}, and γ^{MID}. These parameters need to satisfy the following requirements:

$$C_0 \geq 1,\ C_0 \geq 8D,\ C_0 \geq 4\gamma^{\text{TOP}},\ C_0 \geq 4D^2C_1/Z,\ \gamma^{\text{TOP}} \geq C_1/Z,\ \text{and}\ \gamma^{\text{MID}} \geq 2D^2C_1/Z.$$

As a result, we set

$$C_0 = \max\left\{1, 8D, 4\gamma^{\text{TOP}}, 4D^2C_1\right\},\ \gamma^{\text{TOP}} = C_1,\ \gamma^{\text{MID}} = 2D^2C_1, \tag{A.9}$$

where $Z = \max\{GD + \gamma^{\text{MID}}D^2, 1 + \gamma^{\text{MID}}D^2 + 2\gamma^{\text{TOP}}\}$ and $C_1 = 128(D^2L^2 + G^2)$. ■

Appendix B. Omitted Proofs for Section 4

In this section, we provide the omitted details for Section 4, including the proofs of Lemma 6, the correctness of Lemma 6 under Assumption 5, and Theorem 2.

B.1 Proof of Lemma 6

Proof By inserting intermediate terms, we have

$$\begin{aligned}\bar{V}_T \leq\ & 4\sum_{t=2}^{T}\|\nabla f_t(\mathbf{x}_t) - \nabla f_t(\mathbf{x}_{t,i^\star})\|^2 + 4\sum_{t=2}^{T}\|\nabla f_t(\mathbf{x}_{t,i^\star}) - \nabla f_{t-1}(\mathbf{x}_{t,i^\star})\|^2 \\ & + 4\sum_{t=2}^{T}\|\nabla f_{t-1}(\mathbf{x}_{t,i^\star}) - \nabla f_{t-1}(\mathbf{x}_{t-1,i^\star})\|^2 + 4\sum_{t=2}^{T}\|\nabla f_{t-1}(\mathbf{x}_{t-1,i^\star}) - \nabla f_{t-1}(\mathbf{x}_{t-1})\|^2 \\ \overset{(4.2)}{\leq}\ & 8L\sum_{t=2}^{T}\mathcal{D}_{f_t}(\mathbf{x}_{t,i^\star}, \mathbf{x}_t) + 4V_T + 4L^2\sum_{t=2}^{T}\|\mathbf{x}_{t,i^\star} - \mathbf{x}_{t-1,i^\star}\|^2 + 8L\sum_{t=2}^{T}\mathcal{D}_{f_{t-1}}(\mathbf{x}_{t-1,i^\star}, \mathbf{x}_{t-1}) \\ \leq\ & 4V_T + 16L\sum_{t=1}^{T}\mathcal{D}_{f_t}(\mathbf{x}_{t,i^\star}, \mathbf{x}_t) + 4L^2\sum_{t=2}^{T}\|\mathbf{x}_{t,i^\star} - \mathbf{x}_{t-1,i^\star}\|^2,\end{aligned} \tag{B.1}$$

where the first step introduces intermediate terms $\nabla f_t(\mathbf{x}_{t,i^\star})$, $\nabla f_{t-1}(\mathbf{x}_{t,i^\star})$, and $\nabla f_{t-1}(\mathbf{x}_{t-1,i^\star})$, the second step uses Proposition 1 and the standard smoothness Assumption 3, and the last step combines two summations into one by shifting the indexes of t. ■

B.2 Proof about Relaxed Smoothness

In this part, we show that Lemma 6 also holds under the relaxed Assumption 5. To begin with, we present the following lemma, which shows that Assumption 5 is a sufficient condition for Eq. (4.2) on $\mathcal{X}$. Using Lemma 7, we can see that Lemma 6 also holds under Assumption 5.

Lemma 7. *Under Assumption 5, for any online function $f(\cdot)$ satisfying Assumption 5, it holds that $\|\nabla f(\mathbf{x}) - \nabla f(\mathbf{y})\|^2 \le 2L\mathcal{D}_f(\mathbf{y}, \mathbf{x})$ for any $\mathbf{x}, \mathbf{y} \in \mathcal{X}$.*

Proof To begin with, we present the self-bounding property (Srebro et al., 2010), which is useful in proving our result — if a function $f : \mathbb{R}^d \to \mathbb{R}$ is L-smooth and bounded from below, then for any $\mathbf{x} \in \mathbb{R}^d$, it holds that

$$\|\nabla f(\mathbf{x})\|^2 \le 2L \left(f(\mathbf{x}) - \inf_{\mathbf{y} \in \mathbb{R}^d} f(\mathbf{y}) \right). \tag{B.2}$$

Next, we aim to prove that if we only need (B.2) on a bounded domain $\mathcal{X}$, we require smoothness only on a slightly larger domain than $\mathcal{X}$. To see this, we delve into the proof of the self-bounding property. Specifically, for any $\mathbf{x}, \mathbf{v} \in \mathbb{R}^d$, it holds that

$$\langle -\nabla f(\mathbf{x}), \mathbf{v} \rangle - \frac{L}{2}\|\mathbf{v}\|^2 \le f(\mathbf{x}) - f(\mathbf{x} + \mathbf{v}) \le f(\mathbf{x}) - \inf_{\mathbf{y} \in \mathbb{R}^d} f(\mathbf{y}),$$

where the first step requires smoothness on $\mathbf{x}$ and $\mathbf{x} + \mathbf{v}$. Consequently, by taking maximization over $\mathbf{v}$, it holds that

$$f(\mathbf{x}) - \inf_{\mathbf{y} \in \mathbb{R}^d} f(\mathbf{y}) \ge \sup_{\mathbf{v} \in \mathbb{R}^d} \langle -\nabla f(\mathbf{x}), \mathbf{v} \rangle - \frac{L}{2}\|\mathbf{v}\|^2 = \frac{1}{2L}\|\nabla f(\mathbf{x})\|^2,$$

which leads to the self-bounding property (B.2) by taking $\mathbf{v} = -\frac{1}{L}\nabla f(\mathbf{x})$. The above proof is from Theorem 4.23 of Orabona (2019). This means that for the self-bounding property, we only require the smoothness to hold for any $\mathbf{x} \in \mathcal{X}$ and $\mathbf{x} - \frac{1}{L}\nabla f(\mathbf{x})$. Under Assumption 4, this can be satisfied by requiring smoothness on a slightly larger domain than $\mathcal{X}$, namely, $\mathcal{X}_+ \triangleq \{\mathbf{x} + \boldsymbol{b} \mid \mathbf{x} \in \mathcal{X}, \boldsymbol{b} \in G/L \cdot \mathbb{B}\}$.

Now we are ready to prove the final result. To begin with, we define a surrogate function of $g(\mathbf{x}) \triangleq f(\mathbf{x}) - \langle \nabla f(\mathbf{x}_0), \mathbf{x} \rangle$ for any $\mathbf{x} \in \mathcal{X}$, where $\mathbf{x}_0 \in \mathcal{X}$. Due to the above property we have just proved, by requiring smoothness on $\mathcal{X}_+$, we have

$$\|\nabla g(\mathbf{x})\|^2 \le 2L \left(g(\mathbf{x}) - \inf_{\mathbf{y} \in \mathbb{R}^d} g(\mathbf{y}) \right).$$

Denoting by $\mathbf{y}^\star \in \arg\min_{\mathbf{y} \in \mathbb{R}^d} g(\mathbf{y})$, the above inequality equals to

$$\begin{aligned} \|\nabla f(\mathbf{x}) - \nabla f(\mathbf{x}_0)\|^2 &\le 2L\left(f(\mathbf{x}) - \langle \nabla f(\mathbf{x}_0), \mathbf{x} \rangle - f(\mathbf{y}^\star) + \langle \nabla f(\mathbf{x}_0), \mathbf{y}^\star \rangle\right) \\ &= 2L(f(\mathbf{x}) - f(\mathbf{y}^\star) - \langle \nabla f(\mathbf{x}_0), \mathbf{x} - \mathbf{y}^\star \rangle), \end{aligned}$$

due to the definition of $g(\cdot)$. The proof using the self-bounding property is from Theorem 2.1.5 of Nesterov (2018). Finally, we note that $g(\cdot)$ is minimized at $\mathbf{y}^\star = \mathbf{x}_0$, leading to $\|\nabla f(\mathbf{x}) - \nabla f(\mathbf{x}_0)\|^2 \le 2L\mathcal{D}_f(\mathbf{x}_0, \mathbf{x})$ for any $\mathbf{x}, \mathbf{x}_0 \in \mathcal{X}$, which finishes the proof. ■

B.3 Proof of Theorem 2

Proof For simplicity, we denote by $\mathbf{g}_t \triangleq \nabla f_t(\mathbf{x}_t)$. We start by decomposing the total regret into the meta regret and the base regret. We then analyze the meta regret separately, followed by tailored proofs for different classes of loss functions.

To start with, the meta empirical gradient variation $\bar{V}_T$ can be bounded as

$$\begin{aligned}
\bar{V}_T &= \sum_{t=2}^{T} \|\nabla f_t(\mathbf{x}_t) - \nabla f_{t-1}(\mathbf{x}_{t-1})\|^2 \\
&\le 4\sum_{t=2}^{T} \|\nabla f_t(\mathbf{x}_t) - \nabla f_t(\mathbf{x}_{t,i})\|^2 + 4\sum_{t=2}^{T} \|\nabla f_t(\mathbf{x}_{t,i}) - \nabla f_{t-1}(\mathbf{x}_{t,i})\|^2 \\
&\quad + 4\sum_{t=2}^{T} \|\nabla f_{t-1}(\mathbf{x}_{t,i}) - \nabla f_{t-1}(\mathbf{x}_{t-1,i})\|^2 + 4\sum_{t=2}^{T} \|\nabla f_{t-1}(\mathbf{x}_{t-1,i}) - \nabla f_{t-1}(\mathbf{x}_{t-1})\|^2 \\
&\le 8L\sum_{t=2}^{T} \mathcal{D}_{f_t}(\mathbf{x}_{t,i}, \mathbf{x}_t) + 4V_T + 4L^2\sum_{t=2}^{T} \|\mathbf{x}_{t,i} - \mathbf{x}_{t-1,i}\|^2 + 8L\sum_{t=2}^{T} \mathcal{D}_{f_t}(\mathbf{x}_{t-1,i}, \mathbf{x}_{t-1}) \\
&\le 4V_T + 4L^2 S_{T,i}^{\mathbf{x}} + 16L\sum_{t=1}^{T} \mathcal{D}_{f_t}(\mathbf{x}_{t,i}, \mathbf{x}_t). \qquad \text{(B.3)}
\end{aligned}$$

Similarly, we denote by $\bar{V}_{T,i} = \sum_{t=2}^{T} \|\nabla f_t(\mathbf{x}_{t,i}) - \nabla f_{t-1}(\mathbf{x}_{t-1,i})\|^2$ the empirical gradient variation of the i-th expert, for $i \in [N]$. Then $\bar{V}_{T,i}$ can be bounded as

$$\begin{aligned}
\bar{V}_{T,i} &\le 3\sum_{t=2}^{T} \|\nabla f_t(\mathbf{x}_{t,i}) - \nabla f_t(\mathbf{x}^\star)\|^2 + 3\sum_{t=2}^{T} \|\nabla f_t(\mathbf{x}^\star) - \nabla f_{t-1}(\mathbf{x}^\star)\|^2 \\
&\quad + 3\sum_{t=2}^{T} \|\nabla f_{t-1}(\mathbf{x}^\star) - \nabla f_{t-1}(\mathbf{x}_{t-1,i})\|^2 \\
&\le 6L\sum_{t=2}^{T} \mathcal{D}_{f_t}(\mathbf{x}^\star, \mathbf{x}_{t,i}) + 3V_T + 6L\sum_{t=2}^{T} \mathcal{D}_{f_{t-1}}(\mathbf{x}^\star, \mathbf{x}_{t-1,i}) \\
&\le 3V_T + 12L\sum_{t=1}^{T} \mathcal{D}_{f_t}(\mathbf{x}^\star, \mathbf{x}_{t,i}). \qquad \text{(B.4)}
\end{aligned}$$

Regret Decomposition. For λ-*strongly convex* functions, we decompose the regret as

$$\begin{aligned}
\textsc{Reg}_T \le\ & \underbrace{\sum_{t=1}^{T}\langle \mathbf{g}_t, \mathbf{x}_t - \mathbf{x}_{t,i^\star}\rangle - \frac{\lambda}{2}\sum_{t=1}^{T}\|\mathbf{x}_t - \mathbf{x}_{t,i^\star}\|^2}_{\textsc{Meta-Reg}} \\
& + \underbrace{\sum_{t=1}^{T}\langle \nabla f_t(\mathbf{x}_{t,i^\star}), \mathbf{x}_{t,i^\star} - \mathbf{x}^\star\rangle - \frac{\lambda_{i^\star}}{4}\sum_{t=1}^{T}\|\mathbf{x}^\star - \mathbf{x}_{t,i^\star}\|^2}_{\textsc{Base-Reg}} - \frac{1}{2}\sum_{t=1}^{T}\mathcal{D}_{f_t}(\mathbf{x}^\star, \mathbf{x}_{t,i^\star}), \qquad \text{(B.5)}
\end{aligned}$$

where $\lambda_{i^\star} \le \lambda \le 2\lambda_{i^\star}$. For α-*exp-concave* functions, we decompose the regret as

$$\begin{aligned}\textsc{Reg}_T \le\ & \underbrace{\sum_{t=1}^T \langle \mathbf{g}_t, \mathbf{x}_t - \mathbf{x}_{t,i^\star}\rangle - \frac{\alpha}{2}\sum_{t=1}^T \langle \mathbf{g}_t, \mathbf{x}_t - \mathbf{x}_{t,i^\star}\rangle^2}_{\textsc{Meta-Reg}} \\ & + \underbrace{\sum_{t=1}^T \langle \nabla f_t(\mathbf{x}_{t,i^\star}), \mathbf{x}_{t,i^\star} - \mathbf{x}^\star\rangle - \frac{\alpha_{i^\star}}{4}\sum_{t=1}^T \langle \nabla f_t(\mathbf{x}_{t,i^\star}), \mathbf{x}^\star - \mathbf{x}_{t,i^\star}\rangle^2}_{\textsc{Base-Reg}} - \frac{1}{2}\sum_{t=1}^T \mathcal{D}_{f_t}(\mathbf{x}^\star, \mathbf{x}_{t,i^\star}),\end{aligned} \tag{B.6}$$

where $\alpha_{i^\star} \le \alpha \le 2\alpha_{i^\star}$. For *convex* functions, we decompose the regret as

$$\begin{aligned}\textsc{Reg}_T &= \sum_{t=1}^T f_t(\mathbf{x}_t) - \sum_{t=1}^T f_t(\mathbf{x}_{t,i^\star}) + \sum_{t=1}^T f_t(\mathbf{x}_{t,i^\star}) - \sum_{t=1}^T f_t(\mathbf{x}^\star) \qquad \text{(B.7)} \\ &= \underbrace{\sum_{t=1}^T \langle \mathbf{g}_t, \mathbf{x}_t - \mathbf{x}_{t,i^\star}\rangle}_{\textsc{Meta-Reg}} + \underbrace{\sum_{t=1}^T \langle \nabla f_t(\mathbf{x}_{t,i^\star}), \mathbf{x}_{t,i^\star} - \mathbf{x}^\star\rangle}_{\textsc{Base-Reg}} - \sum_{t=1}^T \mathcal{D}_{f_t}(\mathbf{x}_{t,i^\star}, \mathbf{x}_t) - \sum_{t=1}^T \mathcal{D}_{f_t}(\mathbf{x}^\star, \mathbf{x}_{t,i^\star}).\end{aligned}$$

Meta Regret Analysis. We adopt Optimistic-Adapt-ML-Prod (Wei et al., 2016) as the meta learner, and present its regret analysis below for self-containedness.

Lemma 8 (Theorem 3.4 of Wei et al. (2016))**.** *Denote by $\boldsymbol{p}_t \in \Delta_N$ the algorithm's weights, $\boldsymbol{\ell}_t \in [0,1]^N$ the loss vector, and $m_{t,i}$ the optimism. With the learning rate in (4.7), the regret of Optimistic-Adapt-ML-Prod (4.5) with respect to any expert $i \in [N]$ satisfies*

$$\sum_{t=1}^T \langle \boldsymbol{\ell}_t, \boldsymbol{p}_t - \boldsymbol{e}_i\rangle \le C_0 \sqrt{1 + \sum_{t=1}^T (r_{t,i} - m_{t,i})^2} + C_2,$$

where $\boldsymbol{e}_i$ is the i-th standard basis vector, $C_0 = \sqrt{\log N} + \log(1 + \frac{N}{e}(1 + \log(T+1)))/\sqrt{\log N}$, and $C_2 = \frac{1}{4}(\log N + \log(1 + \frac{N}{e}(1 + \log(T+1)))) + 2\sqrt{\log N} + 16 \log N$.

Here we adopt $\ell_{t,i} = \frac{1}{2GD}\langle \mathbf{g}_t, \mathbf{x}_{t,i}\rangle + \frac{1}{2} \in [0,1]$ such that $\langle \boldsymbol{\ell}_t, \boldsymbol{p}_t - \boldsymbol{e}_i\rangle = \frac{1}{2GD}\langle \mathbf{g}_t, \mathbf{x}_t - \mathbf{x}_{t,i}\rangle$. Besides, since the number of base learners $N = \mathcal{O}(\log T)$ as explained in Section 2, the constants C_0 and C_2 are in the order of $\mathcal{O}(\log \log T)$ and can be ignored, following previous convention (Luo and Schapire, 2015; Gaillard et al., 2014).

For λ-*strongly convex* functions, according to Eq. (4.6), we have $m_{t,i} = 0$ where $\lambda_i \in \mathcal{H}^{\text{sc}}$. By Lemma 8, the meta regret in (B.5) can be bounded as

$$\begin{aligned}\textsc{Meta-Reg} &= 2GD \sum_{t=1}^T \langle \boldsymbol{\ell}_t, \boldsymbol{p}_t - \boldsymbol{e}_i\rangle - \frac{\lambda}{2}\sum_{t=1}^T \|\mathbf{x}_t - \mathbf{x}_{t,i^\star}\|^2 \\ &\le C_0 \sqrt{4G^2D^2 + \sum_{t=1}^T \langle \mathbf{g}_t, \mathbf{x}_t - \mathbf{x}_{t,i^\star}\rangle^2} - \frac{\lambda}{2}\sum_{t=1}^T \|\mathbf{x}_t - \mathbf{x}_{t,i^\star}\|^2 + 2GDC_2 \\ &\le C_0 \sqrt{4G^2D^2 + G^2\sum_{t=1}^T \|\mathbf{x}_t - \mathbf{x}_{t,i^\star}\|^2} - \frac{\lambda}{2}\sum_{t=1}^T \|\mathbf{x}_t - \mathbf{x}_{t,i^\star}\|^2 + 2GDC_2\end{aligned}$$

$$\leq \mathcal{O}(C_3) + \left(\frac{C_0 G^2}{2C_3} - \frac{\lambda}{2}\right) \sum_{t=1}^{T} \|\mathbf{x}_t - \mathbf{x}_{t,i^\star}\|^2, \tag{B.8}$$

where the last step omits the ignorable additive C_0 or C_2 terms and is due to AM-GM inequality (Lemma 18). C_3 is a constant to be specified.

For α-*exp-concave* functions, according to Eq. (4.6), we have $m_{t,i} = 0$ where $\alpha_i \in \mathcal{H}^{\text{exp}}$. By Lemma 8, the meta regret in (B.6) can be bounded as

$$\begin{aligned}
\textsc{Meta-Reg} &\leq C_0 \sqrt{4G^2D^2 + \sum_{t=1}^{T} \langle \mathbf{g}_t, \mathbf{x}_t - \mathbf{x}_{t,i^\star} \rangle^2} - \frac{\alpha}{2} \sum_{t=1}^{T} \langle \mathbf{g}_t, \mathbf{x}_t - \mathbf{x}_{t,i^\star} \rangle^2 + 2GDC_2 \\
&\leq \mathcal{O}(C_4) + \left(\frac{C_0}{2C_4} - \frac{\alpha}{2}\right) \sum_{t=1}^{T} \langle \mathbf{g}_t, \mathbf{x}_t - \mathbf{x}_{t,i^\star} \rangle^2, \qquad \text{(B.9)}
\end{aligned}$$

where the last step omits the ignorable additive C_0 or C_2 terms and is due to AM-GM inequality (Lemma 18). C_4 is a constant to be specified.

For *convex* functions, according to Eq. (4.6), we have $m_{t,i} = \langle \mathbf{g}_{t-1}, \mathbf{x}_t - \mathbf{x}_{t,i} \rangle / (2GD)$ for the convex base learner. As explained in Section 4, although $\mathbf{x}_t$ is unknown for now, we only require the scalar value of $\langle \mathbf{g}_{t-1}, \mathbf{x}_t \rangle$. Denoting by $z = \langle \mathbf{g}_{t-1}, \mathbf{x}_t \rangle$, it actually forms a fixed-point problem of $z = \langle \mathbf{g}_{t-1}, \mathbf{x}_t(z) \rangle$, where $\mathbf{x}_t$ is a function of z since $\mathbf{x}_t$ depends on $p_{t,i}$, $p_{t,i}$ relies on $m_{t,i}$, and $m_{t,i}$ depends on z. Such a one-dimensional fixed-point problem can be solved with an $\mathcal{O}(1/T)$ approximation error through $\mathcal{O}(\log T)$ binary searches, and aggregating the approximate error over the whole time horizon will only incur an additive constant to the final regret. As a result, such an optimism setup is valid. Consequently, the meta regret in (B.7) can be bounded as

$$\begin{aligned}
\textsc{Meta-Reg} &\leq C_0 \sqrt{4G^2D^2 + \sum_{t=1}^{T} \langle \mathbf{g}_t - \mathbf{g}_{t-1}, \mathbf{x}_t - \mathbf{x}_{t,i^\star} \rangle^2} + 2GDC_2 \leq C_0 \sqrt{1 + D^2 \bar{V}_T} + C_2 \\
&\overset{\text{(B.3)}}{\leq} C_0 \sqrt{4G^2D^2 + 4D^2V_T + 4L^2D^2 S^{\mathbf{x}}_{T,i^\star} + 16LD^2 \sum_{t=1}^{T} \mathcal{D}_{f_t}(\mathbf{x}_{t,i^\star}, \mathbf{x}_t)} + 2GDC_2 \\
&\leq \mathcal{O}(\sqrt{V_T}) + C_0 \sqrt{4L^2D^2 S^{\mathbf{x}}_{T,i^\star} + 16LD^2 \sum_{t=1}^{T} \mathcal{D}_{f_t}(\mathbf{x}_{t,i^\star}, \mathbf{x}_t)} \\
&\leq \mathcal{O}(\sqrt{V_T}) + \mathcal{O}(C_5) + \frac{C_0}{2C_5} \sum_{t=1}^{T} \mathcal{D}_{f_t}(\mathbf{x}_{t,i^\star}, \mathbf{x}_t) + \frac{LC_0}{8C_5} S^{\mathbf{x}}_{T,i^\star}, \qquad \text{(B.10)}
\end{aligned}$$

where the second step adopts Assumption 1. Note that C_5 is used to ensure the positive Bregman divergence term to be canceled and will be specified in the end.

Base Regret Analysis. For λ-*strongly convex* functions, according to Lemma 19, the base regret can be bounded by

$$\textsc{Base-Reg} \leq \frac{16G^2}{\lambda_{i^\star}} \log\left(1 + \lambda_{i^\star} G^2 \bar{V}_{T,i^\star}\right) + \mathcal{O}(1)$$

$$\leq \frac{16G^2}{\lambda_{i^\star}} \log\left(1 + 3\lambda_{i^\star} G^2 V_T + 12\lambda_{i^\star} L G^2 \sum_{t=1}^T \mathcal{D}_{f_t}(\mathbf{x}^\star, \mathbf{x}_{t,i^\star})\right) + \mathcal{O}(1)$$

$$\leq \mathcal{O}\left(\frac{1}{\lambda}\log V_T\right) + \mathcal{O}(\log C_6) + \frac{192LG^2}{C_6}\sum_{t=1}^T \mathcal{D}_{f_t}(\mathbf{x}^\star, \mathbf{x}_{t,i^\star}), \tag{B.11}$$

where the last step uses $\log(1+x) \leq x$ for any $x > -1$.

For α-*exp-concave* functions, by Lemma 20, the base regret can be bounded by

$$\begin{aligned}\textsc{Base-Reg} &\leq \frac{16d}{\alpha_{i^\star}}\log\left(1 + \frac{\alpha_{i^\star}}{d}\bar{V}_{T,i^\star}\right) + \mathcal{O}(1)\\ &\leq \frac{16d}{\alpha_{i^\star}}\log\left(1 + \frac{3\alpha_{i^\star}}{d}V_T + \frac{12\alpha_{i^\star}L}{d}\sum_{t=1}^T \mathcal{D}_{f_t}(\mathbf{x}^\star, \mathbf{x}_{t,i^\star})\right) + \mathcal{O}(1)\\ &\leq \mathcal{O}\left(\frac{d}{\alpha}\log V_T\right) + \mathcal{O}(\log C_7) + \frac{192L}{C_7}\sum_{t=1}^T \mathcal{D}_{f_t}(\mathbf{x}^\star, \mathbf{x}_{t,i^\star}),\end{aligned} \tag{B.12}$$

where the last step is by $\log(1+x) \leq x$.

For *convex* functions, by Lemma 21, the base regret can be bounded by

$$\begin{aligned}\textsc{Base-Reg} \leq& 5D\sqrt{1 + \bar{V}_{T,i^\star}} - \frac{\kappa}{4}S^{\mathbf{x}}_{T,i^\star} + \mathcal{O}(1)\\ \leq& 5D\sqrt{1 + 3V_T + 12L\sum_{t=1}^T \mathcal{D}_{f_t}(\mathbf{x}^\star, \mathbf{x}_{t,i^\star})} - \frac{\kappa}{4}S^{\mathbf{x}}_{T,i^\star} + \mathcal{O}(1)\\ \leq& \mathcal{O}(\sqrt{V_T}) + \mathcal{O}(C_8) + \frac{5DL}{2C_8}\sum_{t=1}^T \mathcal{D}_{f_t}(\mathbf{x}^\star, \mathbf{x}_{t,i^\star}) - \frac{\kappa}{4}S^{\mathbf{x}}_{T,i^\star}.\end{aligned} \tag{B.13}$$

Overall Regret Analysis. For λ-*strongly convex* functions, plugging Eq. (B.8) and Eq. (B.11) into Eq. (B.5), we obtain

$$\begin{aligned}\textsc{Reg}_T \leq& \mathcal{O}\left(\frac{1}{\lambda}\log V_T\right) + \mathcal{O}(C_3 + \log C_6) + \left(\frac{192LG^2}{C_6} - \frac{1}{2}\right)\sum_{t=1}^T \mathcal{D}_{f_t}(\mathbf{x}^\star, \mathbf{x}_{t,i^\star})\\ &+ \left(\frac{C_0 D^2}{2C_3} - \frac{\lambda}{2}\right)\sum_{t=1}^T \|\mathbf{x}_t - \mathbf{x}_{t,i^\star}\|^2 \leq \mathcal{O}\left(\frac{1}{\lambda}\log V_T\right),\end{aligned}$$

by choosing $C_3 = 2C_0/\lambda$ and $C_6 = 384LG^2$.

For α-*exp-concave* functions, plugging Eq. (B.9) and Eq. (B.12) into Eq. (B.6), we obtain

$$\begin{aligned}\textsc{Reg}_T \leq& \mathcal{O}\left(\frac{d}{\alpha}\log V_T\right) + \mathcal{O}(C_4 + \log C_7) + \left(\frac{192L}{C_7} - \frac{1}{2}\right)\sum_{t=1}^T \mathcal{D}_{f_t}(\mathbf{x}^\star, \mathbf{x}_{t,i^\star})\\ &+ \left(\frac{C_0}{2C_4} - \frac{\alpha}{2}\right)\sum_{t=1}^T \langle \mathbf{g}_t, \mathbf{x}_t - \mathbf{x}_{t,i^\star}\rangle^2 \leq \mathcal{O}\left(\frac{d}{\alpha}\log V_T\right),\end{aligned}$$

by choosing $C_4 = 2C_0/\alpha$ and $C_7 = 384L$.

For *convex* functions, plugging Eq. (B.10) and Eq. (B.13) into Eq. (B.7), we obtain

$$\begin{aligned}\textsc{Reg}_T \leq{}& \mathcal{O}(\sqrt{V_T}) + \mathcal{O}(C_5 + C_8) + \left(\frac{5DL}{2C_8} - 1\right)\sum_{t=1}^T \mathcal{D}_{f_t}(\mathbf{x}^\star, \mathbf{x}_{t,i^\star}) + \left(\frac{LC_0}{8C_5} - \frac{\kappa}{4}\right) S^{\mathbf{x}}_{T,i^\star}\\ &+ \left(\frac{C_0}{2C_5} - 1\right)\sum_{t=1}^T \mathcal{D}_{f_t}(\mathbf{x}_{t,i^\star}, \mathbf{x}_t) \leq \mathcal{O}(\sqrt{V_T}),\end{aligned}$$

by choosing $C_5 = C_0$, $C_8 = 5DL$, and $\kappa \geq L/2$.

Note that the constants $C_3, C_4, C_5, C_6, C_7, C_8$ appear only in the analysis, and hence our choices of them are feasible. ■

Appendix C. Omitted Proofs for Section 5

In this section, we provide the proofs for the regret guarantees of the efficient algorithms presented in Section 5, including the details of base learners' updates on surrogates, the proofs of Proposition 2, Theorem 3, and Theorem 4.

C.1 Details of Base Learners' Update

Base Learners. To begin with, we duplicate the candidate coefficient pool (2.5) for both the exp-concave coefficient α and the strongly convex coefficient λ, denoted by $\mathcal{H}^{\exp} \triangleq \mathcal{H}$ and $\mathcal{H}^{\mathrm{sc}} \triangleq \mathcal{H}$. Consequently, denoting by $N^{\exp} = N^{\mathrm{sc}} \triangleq |\mathcal{H}|$ the size of candidate pool, for each $\alpha_i \in \mathcal{H}^{\exp}$ and $\lambda_j \in \mathcal{H}^{\mathrm{sc}}$, where $i \in [N^{\exp}]$ and $j \in [N^{\mathrm{sc}}]$, we define corresponding groups of base learners for optimizing exp-concave and strongly convex functions. Specifically, for *α-exp-concave* functions, we define a group of base learners $\{\mathcal{B}_i^{\exp}\}_{i\in[N]}$, where the i-th base learner runs the algorithm below:

$$\begin{aligned}\mathbf{x}_{t,i} &= \operatorname*{arg\,min}_{\mathbf{x}\in\mathcal{X}} \left\{\langle \nabla h^{\exp}_{t-1,i}(\mathbf{x}_{t-1,i}), \mathbf{x}\rangle + \mathcal{D}_{\psi_{t,i}}(\mathbf{x}, \widehat{\mathbf{x}}_{t,i})\right\},\\ \widehat{\mathbf{x}}_{t+1,i} &= \operatorname*{arg\,min}_{\mathbf{x}\in\mathcal{X}} \left\{\langle \nabla h^{\exp}_{t,i}(\mathbf{x}_{t,i}), \mathbf{x}\rangle + \mathcal{D}_{\psi_{t,i}}(\mathbf{x}, \widehat{\mathbf{x}}_{t,i})\right\},\end{aligned} \tag{C.1}$$

where $\psi_{t,i}(\mathbf{x}) \triangleq \frac{1}{2}\mathbf{x}^\top U_{t,i}\mathbf{x}$, $U_{t,i} = (1 + \frac{\alpha_i G^2}{2})I + \frac{\alpha_i}{2}\sum_{s=1}^{t-1} \nabla h^{\exp}_{s,i}(\mathbf{x}_{s,i}) h^{\exp}_{s,i}(\mathbf{x}_{s,i})^\top$, α_i is the i-th element in $\mathcal{H}^{\exp}$, and $h^{\exp}_{t,i}(\cdot)$ is a surrogate loss function for $\mathcal{B}_i^{\exp}$, defined as

$$h^{\exp}_{t,i}(\mathbf{x}) \triangleq \langle \nabla f_t(\mathbf{x}_t), \mathbf{x}\rangle + \frac{\alpha_i}{4}\langle \nabla f_t(\mathbf{x}_t), \mathbf{x} - \mathbf{x}_t\rangle^2.$$

Similarly, for *λ-strongly convex* functions, we define a group of base learners $\{\mathcal{B}_i^{\mathrm{sc}}\}_{i\in[N^{\mathrm{sc}}]}$, where the i-th base learner runs the algorithm below:

$$\mathbf{x}_{t,i} = \Pi_{\mathcal{X}}[\widehat{\mathbf{x}}_{t,i} - \eta_{t,i}\nabla h^{\mathrm{sc}}_{t-1,i}(\mathbf{x}_{t-1,i})], \quad \widehat{\mathbf{x}}_{t+1,i} = \Pi_{\mathcal{X}}[\widehat{\mathbf{x}}_{t,i} - \eta_{t,i}\nabla h^{\mathrm{sc}}_{t,i}(\mathbf{x}_{t,i})], \tag{C.2}$$

where $\eta_{t,i} = 2/(1+\lambda_i t)$, λ_i is the i-th element in $\mathcal{H}^{\mathrm{sc}}$, and $h^{\mathrm{sc}}_{t,i}(\cdot)$ is a surrogate loss function for $\mathcal{B}_i^{\mathrm{sc}}$, defined as

$$h^{\mathrm{sc}}_{t,i}(\mathbf{x}) \triangleq \langle \nabla f_t(\mathbf{x}_t), \mathbf{x}\rangle + \frac{\lambda_i}{4}\|\mathbf{x} - \mathbf{x}_t\|^2.$$

For *convex* functions, we only have to define one base learner $\mathcal{B}^{\text{c}}$, which updates as

$$\mathbf{x}_{t,i} = \Pi_{\mathcal{X}}[\widehat{\mathbf{x}}_{t,i} - \eta_{t,i}\nabla f_{t-1}(\mathbf{x}_{t-1})], \quad \widehat{\mathbf{x}}_{t+1,i} = \Pi_{\mathcal{X}}[\widehat{\mathbf{x}}_{t,i} - \eta_{t,i}\nabla f_t(\mathbf{x}_t)], \tag{C.3}$$

where $\eta_{t,i} = \min\{D/\sqrt{1 + \sum_{s=2}^{t-1}\|\nabla f_t(\mathbf{x}_{t,i}) - \nabla f_{t-1}(\mathbf{x}_{t-1,i})\|^2}, 1\}$. Finally, we conclude the configurations of base learners. Specifically, we deploy

$$\{\mathcal{B}_i\}_{i\in[N]} \triangleq \{\mathcal{B}_i^{\text{exp}}\}_{i\in[N^{\text{exp}}]} \cup \{\mathcal{B}_i^{\text{sc}}\}_{i\in[N^{\text{sc}}]} \cup \{\mathcal{B}^{\text{c}}\}, \text{ where } N \triangleq N^{\text{exp}} + N^{\text{sc}} + 1, \tag{C.4}$$

as the total set of base learners.

C.2 Proof of Proposition 2

Proof For simplicity, we use $\mathbf{g}_t \triangleq \nabla f_t(\mathbf{x}_t)$. For the meta regret, we use Adapt-ML-Prod (Gaillard et al., 2014) to optimize the linear loss $\boldsymbol{\ell}_t = (\ell_{t,1}, \ldots, \ell_{t,N})$, where $\ell_{t,i} \triangleq \frac{1}{2GD}\langle \mathbf{g}_t, \mathbf{x}_{t,i}\rangle + \frac{1}{2} \in [0,1]$, and obtain the following second-order bound by Corollary 4 of Gaillard et al. (2014),

$$\begin{aligned}\sum_{t=1}^T \langle \mathbf{g}_t, \mathbf{x}_t - \mathbf{x}_{t,i^\star}\rangle &= 2GD\sum_{t=1}^T \langle \boldsymbol{\ell}_t, \boldsymbol{p}_t - \boldsymbol{e}_i\rangle \\ &\lesssim \sqrt{\log\log T \sum_{t=1}^T \langle \boldsymbol{\ell}_t, \boldsymbol{p}_t - \boldsymbol{e}_i\rangle^2} \lesssim \sqrt{\log\log T \sum_{t=1}^T \langle \mathbf{g}_t, \mathbf{x}_t - \mathbf{x}_{t,i^\star}\rangle^2}.\end{aligned}$$

For α-*exp-concave* functions, it holds that

$$\begin{aligned}\textsc{Meta-Reg} &\lesssim \sqrt{\log\log T \sum_{t=1}^T \langle \mathbf{g}_t, \mathbf{x}_t - \mathbf{x}_{t,i^\star}\rangle^2} - \frac{\alpha_{i^\star}}{2}\sum_{t=1}^T \langle \mathbf{g}_t, \mathbf{x}_t - \mathbf{x}_{t,i^\star}\rangle^2 \\ &\le \frac{\log\log T}{2\alpha_{i^\star}} \le \frac{\log\log T}{\alpha}, \qquad\qquad (\text{by } \alpha_{i^\star} \le \alpha \le 2\alpha_{i^\star})\end{aligned}$$

where the second step uses AM-GM inequality (Lemma 18) with $a = \alpha_{i^\star}$. To handle the base regret, by optimizing the surrogate loss function $h_{t,i^\star}^{\text{exp}}$ using Online Newton Step (ONS), it holds that

$$\textsc{Base-Reg} \lesssim \frac{dDG_{\text{exp}}}{\alpha_{i^\star}}\log T \le \frac{2dD(G+GD)}{\alpha}\log T,$$

where $G_{\text{exp}} \triangleq \max_{\mathbf{x}\in\mathcal{X}, t\in[T], i\in[N]} \|\nabla h_{t,i}^{\text{exp}}(\mathbf{x})\| \le G + GD$ represents the maximum gradient norm the last step is because $\alpha \le 2\alpha_{i^\star}$. Combining the meta and base regret, the regret can be bounded by $\mathcal{O}(d\log T)$.

For λ-*strongly convex* functions, since it is also $\alpha = \lambda/G^2$ exp-concave under Assumption 2 (Hazan et al., 2007, Section 2.2), the above meta regret analysis is still applicable. To optimize the base regret, by optimizing the surrogate loss function $h_{t,i^\star}^{\text{sc}}$ using Online Gradient Descent (OGD), it holds that

$$\textsc{Base-Reg} \le \frac{G_{\text{sc}}^2}{\lambda_{i^\star}}(1 + \log T) \le \frac{2(G+D)^2}{\lambda}(1 + \log T),$$

where $G_{\mathrm{sc}} \triangleq \max_{\mathbf{x}\in\mathcal{X},t\in[T],i\in[N]} \|\nabla h^{\mathrm{sc}}_{t,i}(\mathbf{x})\| \le G + D$ represents the maximum gradient norm and the last step is because $\lambda \le 2\lambda_{i^\star}$. Thus the overall regret can be bounded by $\mathcal{O}(\log T)$.

For *convex* functions, the meta regret can be bounded by $\mathcal{O}(\sqrt{T\log\log T})$, where the $\log\log T$ factor can be omitted in the $\mathcal{O}(\cdot)$-notation, and the base regret can be bounded by $\mathcal{O}(\sqrt{T})$ using OGD, resulting in an $\mathcal{O}(\sqrt{T})$ regret overall, which completes the proof. ■

C.3 Proof of Theorem 3

Proof Recall that we denote by $\mathbf{g}_t = \nabla f_t(\mathbf{x}_t)$ for simplicity. We first give different regret decompositions, then analyze the meta regret, and finally provide the proofs for different kinds of loss functions. Some abbreviations of the stability terms are defined in (A.1).

Regret Decomposition. For λ-*strongly convex* functions, we have

$$\begin{aligned}\textsc{Reg}_T &\le \sum_{t=1}^T f_t(\mathbf{x}_t) - \sum_{t=1}^T f_t(\mathbf{x}^\star) \le \sum_{t=1}^T \langle \mathbf{g}_t, \mathbf{x}_t - \mathbf{x}^\star\rangle - \frac{\lambda_{i^\star}}{2}\sum_{t=1}^T \|\mathbf{x}_t - \mathbf{x}^\star\|^2 \\ &= \underbrace{\sum_{t=1}^T \langle \mathbf{g}_t, \mathbf{x}_t - \mathbf{x}_{t,i^\star}\rangle - \frac{\lambda_{i^\star}}{2}\|\mathbf{x}_t - \mathbf{x}_{t,i^\star}\|^2}_{\textsc{Meta-Reg}} + \underbrace{\sum_{t=1}^T h^{\mathrm{sc}}_{t,i^\star}(\mathbf{x}_{t,i^\star}) - \sum_{t=1}^T h^{\mathrm{sc}}_{t,i^\star}(\mathbf{x}^\star)}_{\textsc{Base-Reg}},\end{aligned}$$

where the first step is by $\lambda_{i^\star} \le \lambda \le 2\lambda_{i^\star}$ and the last step holds by defining surrogate loss functions $h^{\mathrm{sc}}_{t,i}(\mathbf{x}) \triangleq \langle \mathbf{g}_t, \mathbf{x}\rangle + \frac{\lambda_i}{2}\|\mathbf{x} - \mathbf{x}_t\|^2$. Similarly, for α-*exp-concave* functions, the regret can be upper-bounded by

$$\textsc{Reg}_T \le \underbrace{\sum_{t=1}^T \langle \mathbf{g}_t, \mathbf{x}_t - \mathbf{x}_{t,i^\star}\rangle - \frac{\alpha_{i^\star}}{2}\langle \mathbf{g}_t, \mathbf{x}_t - \mathbf{x}^\star\rangle^2}_{\textsc{Meta-Reg}} + \underbrace{\sum_{t=1}^T h^{\exp}_{t,i^\star}(\mathbf{x}_{t,i^\star}) - \sum_{t=1}^T h^{\exp}_{t,i^\star}(\mathbf{x}^\star)}_{\textsc{Base-Reg}},$$

by defining surrogate loss functions $h^{\exp}_{t,i}(\mathbf{x}) \triangleq \langle \mathbf{g}_t, \mathbf{x}\rangle + \frac{\alpha_i}{2}\langle \mathbf{g}_t, \mathbf{x} - \mathbf{x}_t\rangle^2$.

For *convex* functions, the regret can be decomposed as:

$$\textsc{Reg}_T \le \underbrace{\sum_{t=1}^T \langle \mathbf{g}_t, \mathbf{x}_t - \mathbf{x}_{t,i^\star}\rangle}_{\textsc{Meta-Reg}} + \underbrace{\sum_{t=1}^T \langle \mathbf{g}_t, \mathbf{x}_{t,i^\star} - \mathbf{x}^\star\rangle}_{\textsc{Base-Reg}}.$$

Meta Regret Analysis. The structure of the meta regret analysis in this part parallels that in Appendix A.4. Recall that, we let $V_\star = \sum_{t=2}^T (\ell^{\textsc{mid}}_{t,j^\star,i^\star} - m^{\textsc{mid}}_{t,j^\star,i^\star})^2$ for simplicity.

For λ-*strongly convex* functions, applying Eq. (A.5) with the stability and correction-induced negative terms, the meta regret satisfies

$$\begin{aligned}\textsc{Meta-Reg} &\overset{\text{(A.5)}}{\le} 2ZC_0\log\frac{4N}{3} + \frac{512ZG^2}{\lambda}\log\frac{2^{20}G^2N}{3C_0^2\lambda^2} + \gamma^{\textsc{mid}} S^{\mathbf{x}}_{T,i^\star} - \frac{C_0}{2}S^{\textsc{top}}_T \\ &- \gamma^{\textsc{mid}}\sum_{t=2}^T\sum_{j=1}^M q^{\textsc{top}}_{t,j}\sum_{i=1}^N q^{\textsc{mid}}_{t,j,i}\|\mathbf{x}_{t,i} - \mathbf{x}_{t-1,i}\|^2 - \gamma^{\textsc{top}}\sum_{t=2}^T\sum_{j=1}^M q^{\textsc{top}}_{t,j}\|\boldsymbol{q}^{\textsc{mid}}_{t,j} - \boldsymbol{q}^{\textsc{mid}}_{t-1,j}\|_1^2,\end{aligned} \tag{C.5}$$

using Lemma 13 and requiring $\varepsilon_{j^\star}^{\text{TOP}} \leq \varepsilon_\star^{\text{TOP}} \triangleq \frac{\lambda_{i^\star} Z}{256G^2}$.

For α*-exp-concave* functions, applying Eq. (A.6) with the stability and correction-induced negative terms, the meta regret can be similarly bounded by

$$\begin{aligned}\text{Meta-Reg} &\overset{\text{(A.6)}}{\leq} 2ZC_0 \log \frac{4N}{3} + \frac{512Z}{\alpha} \log \frac{2^{20}N}{3C_0^2\alpha^2} + \gamma^{\text{MID}} S_{T,i^\star}^{\mathbf{x}} - \frac{C_0}{2} S_T^{\text{TOP}} \\ &- \gamma^{\text{MID}} \sum_{t=2}^{T}\sum_{j=1}^{M} q_{t,j}^{\text{TOP}} \sum_{i=1}^{N} q_{t,j,i}^{\text{MID}} \|\mathbf{x}_{t,i} - \mathbf{x}_{t-1,i}\|^2 - \gamma^{\text{TOP}} \sum_{t=2}^{T}\sum_{j=1}^{M} q_{t,j}^{\text{TOP}} \|\boldsymbol{q}_{t,j}^{\text{MID}} - \boldsymbol{q}_{t-1,j}^{\text{MID}}\|_1^2, \end{aligned} \tag{C.6}$$

using Lemma 13 and requiring $\varepsilon_{j^\star}^{\text{TOP}} \leq \varepsilon_\star^{\text{TOP}} \triangleq \frac{\alpha_{i^\star} Z}{512}$.

For *convex* functions, according to Eq. (A.8) in Appendix A.4, the meta regret satisfies

$$\begin{aligned}\text{Meta-Reg} &\leq \sum_{t=1}^{T} \langle \mathbf{g}_t, \mathbf{x}_t - \mathbf{x}_{t,i^\star} \rangle \\ &\leq 2ZC_0 \log \frac{4N}{3} + 32D\sqrt{2V_T \log(512ND^2V_T/Z^2)} + \left(\frac{64G^2}{Z} + \gamma^{\text{MID}}\right) S_{T,i^\star}^{\mathbf{x}} + \frac{C_1}{2Z} S_T^{\mathbf{x}} \qquad \text{(C.7)} \\ &- \frac{C_0}{2} S_T^{\text{TOP}} - \gamma^{\text{MID}} \sum_{t=2}^{T}\sum_{j=1}^{M} q_{t,j}^{\text{TOP}} \sum_{i=1}^{N} q_{t,j,i}^{\text{MID}} \|\mathbf{x}_{t,i} - \mathbf{x}_{t-1,i}\|^2 - \gamma^{\text{TOP}} \sum_{t=2}^{T}\sum_{j=1}^{M} q_{t,j}^{\text{TOP}} \|\boldsymbol{q}_{t,j}^{\text{MID}} - \boldsymbol{q}_{t-1,j}^{\text{MID}}\|_1^2,\end{aligned}$$

where $C_1 = 128(D^2L^2 + G^2)$.

Base Regret Analysis. In this part, we first provide different decompositions of the empirical gradient variation defined on surrogates for strongly convex, exp-concave, and convex functions, respectively, and then analyze the base regret in the corresponding cases.

For λ*-strongly convex* functions, we bound the empirical gradient variation on surrogates, i.e., $\bar{V}_{T,i^\star}^{\text{sc}} \triangleq \sum_{t=2}^{T} \|\nabla h_{t,i^\star}^{\text{sc}}(\mathbf{x}_{t,i^\star}) - \nabla h_{t-1,i^\star}^{\text{sc}}(\mathbf{x}_{t-1,i^\star})\|^2$, by

$$\begin{aligned}\bar{V}_{T,i^\star}^{\text{sc}} &= \sum_{t=2}^{T} \|(\mathbf{g}_t + \lambda_{i^\star}(\mathbf{x}_{t,i^\star} - \mathbf{x}_t)) - (\mathbf{g}_{t-1} + \lambda_{i^\star}(\mathbf{x}_{t-1,i^\star} - \mathbf{x}_{t-1}))\|^2 \\ &\leq 2\sum_{t=2}^{T} \|\mathbf{g}_t - \mathbf{g}_{t-1}\|^2 + 2\lambda_{i^\star}^2 \sum_{t=2}^{T} \|(\mathbf{x}_{t,i^\star} - \mathbf{x}_t) - (\mathbf{x}_{t-1,i^\star} - \mathbf{x}_{t-1})\|^2 \\ &\leq 4V_T + (4 + 4L^2) S_T^{\mathbf{x}} + 4S_{T,i^\star}^{\mathbf{x}}, \qquad \text{(by } \lambda \in [1/T, 1])\end{aligned}$$

where the first step uses the definition of $\nabla h_{t,i}^{\text{sc}}(\mathbf{x}) = \mathbf{g}_t + \lambda_i(\mathbf{x} - \mathbf{x}_t)$. For α*-exp-concave* functions, we control the $\bar{V}_{T,i^\star}^{\exp} \triangleq \sum_{t=2}^{T} \|\nabla h_{t,i^\star}^{\exp}(\mathbf{x}_{t,i^\star}) - \nabla h_{t-1,i^\star}^{\exp}(\mathbf{x}_{t-1,i^\star})\|^2$ as

$$\begin{aligned}\bar{V}_{T,i^\star}^{\exp} &= \sum_{t=2}^{T} \|(\mathbf{g}_t + \alpha_{i^\star}\mathbf{g}_t\mathbf{g}_t^\top(\mathbf{x}_{t,i^\star} - \mathbf{x}_t)) - (\mathbf{g}_{t-1} + \alpha_{i^\star}\mathbf{g}_{t-1}\mathbf{g}_{t-1}^\top(\mathbf{x}_{t-1,i^\star} - \mathbf{x}_{t-1}))\|^2 \\ &\leq 2\sum_{t=2}^{T} \|\mathbf{g}_t - \mathbf{g}_{t-1}\|^2 + 2\alpha_{i^\star}^2 \sum_{t=2}^{T} \|\mathbf{g}_t\mathbf{g}_t^\top(\mathbf{x}_{t,i^\star} - \mathbf{x}_t) - \mathbf{g}_{t-1}\mathbf{g}_{t-1}^\top(\mathbf{x}_{t-1,i^\star} - \mathbf{x}_{t-1})\|^2 \\ &\leq 4V_T + 4L^2 S_T^{\mathbf{x}} + 4D^2 \sum_{t=2}^{T} \|\mathbf{g}_t\mathbf{g}_t^\top - \mathbf{g}_{t-1}\mathbf{g}_{t-1}^\top\|^2 \qquad \text{(by (3.2))}\end{aligned}$$

$$+ 4G^4 \sum_{t=2}^{T} \|(\mathbf{x}_{t,i^\star} - \mathbf{x}_t) - (\mathbf{x}_{t-1,i^\star} - \mathbf{x}_{t-1})\|^2 \qquad \text{(by } \alpha \in [1/T, 1]\text{)}$$
$$\leq C_9 V_T + C_{10} S_T^{\mathbf{x}} + 8G^4 S_{T,i^\star}^{\mathbf{x}},$$

where the first step uses the definition of $\nabla h_{t,i}^{\text{exp}}(\mathbf{x}) = \mathbf{g}_t + \alpha_i \mathbf{g}_t \mathbf{g}_t^\top(\mathbf{x} - \mathbf{x}_t)$ and the last step holds by setting $C_9 = 4 + 32D^2G^2$ and $C_{10} = 4L^2 + 32D^2G^2L^2 + 8G^4$. For *convex* functions, the empirical gradient variation $\bar{V}_{T,i^\star}^{\text{c}} \triangleq \sum_{t=2}^{T} \|\nabla f_t(\mathbf{x}_t) - \nabla f_{t-1}(\mathbf{x}_{t-1})\|^2$ can be bounded by $\bar{V}_{T,i^\star}^{\text{c}} \leq 2V_T + 2L^2 S_T^{\mathbf{x}}$. To conclude, for different curvature types, we provide correspondingly different analysis of the empirical gradient variation on surrogates:

$$\bar{V}_{T,i^\star}^{\{\text{sc},\text{exp},\text{c}\}} \leq \begin{cases} 4V_T + (4 + 4L^2) S_T^{\mathbf{x}} + 4 S_{T,i^\star}^{\mathbf{x}}, & \text{when } \{f_t\}_{t=1}^T \text{ are } \lambda\text{-strongly convex,} \\ C_9 V_T + C_{10} S_T^{\mathbf{x}} + 8G^4 S_{T,i^\star}^{\mathbf{x}}, & \text{when } \{f_t\}_{t=1}^T \text{ are } \alpha\text{-exp-concave,} \\ 2V_T + 2L^2 S_T^{\mathbf{x}}, & \text{when } \{f_t\}_{t=1}^T \text{ are convex.} \end{cases} \tag{C.8}$$

In the following, we analyze the base regret for different curvature types. For *λ-strongly convex* functions, by Lemma 19, the $i^\star$-th base learner guarantees the following:

$$\textsc{Base-Reg} \leq \frac{16G^2}{\lambda_{i^\star}} \log\left(1 + \lambda_{i^\star} \bar{V}_{T,i^\star}^{\text{sc}}\right) + \frac{1}{4}\kappa D^2 - \frac{1}{8}\kappa S_{T,i^\star}^{\mathbf{x}} + \mathcal{O}(1) \tag{C.9}$$
$$\leq \frac{16G^2}{\lambda_{i^\star}} \log(1 + 4\lambda_{i^\star} V_T + (4 + 4L^2)\lambda_{i^\star} S_T^{\mathbf{x}} + 4\lambda_{i^\star} S_{T,i^\star}^{\mathbf{x}}) + \frac{1}{4}\kappa D^2 - \frac{1}{8}\kappa S_{T,i^\star}^{\mathbf{x}}$$
$$\leq \frac{32G^2}{\lambda} \log(1 + 4\lambda V_T) + (64 + 64L^2) G^2 S_T^{\mathbf{x}} + 64G^2 S_{T,i^\star}^{\mathbf{x}} + \frac{1}{4}\kappa D^2 - \frac{1}{8}\kappa S_{T,i^\star}^{\mathbf{x}}, \tag{C.10}$$

where the constant $\mathcal{O}(1)$ is omitted from the second step and the last step is due to $\log(1 + x) \leq x$ for $x \geq 0$.

For *α-exp-concave* functions, by Lemma 20, the $i^\star$-th base learner guarantees:

$$\textsc{Base-Reg} \leq \frac{16d}{\alpha_{i^\star}} \log\left(1 + \frac{\alpha_{i^\star}}{8\kappa d} \bar{V}_{T,i^\star}^{\text{exp}}\right) + \frac{1}{2}\kappa D^2 - \frac{1}{4}\kappa S_{T,i^\star}^{\mathbf{x}} + \mathcal{O}(1) \tag{C.11}$$
$$\leq \frac{16d}{\alpha_{i^\star}} \log\left(1 + \frac{\alpha_{i^\star} C_9}{8\kappa d} V_T + \frac{\alpha_{i^\star} C_{10}}{8\kappa d} S_T^{\mathbf{x}} + \frac{\alpha_{i^\star} G^4}{\kappa d} S_{T,i^\star}^{\mathbf{x}}\right) + \frac{1}{2}\kappa D^2 - \frac{1}{4}\kappa S_{T,i^\star}^{\mathbf{x}} + \mathcal{O}(1)$$
$$\leq \frac{32d}{\alpha} \log\left(1 + \frac{\alpha C_9}{8\kappa d} V_T\right) + \frac{2C_{10}}{\kappa} S_T^{\mathbf{x}} + \left(\frac{16G^4}{\kappa} - \frac{\kappa}{4}\right) S_{T,i^\star}^{\mathbf{x}} + \frac{1}{2}\kappa D^2 + \mathcal{O}(1), \tag{C.12}$$

where the last step is due to $\log(1 + x) \leq x$ for $x \geq 0$.

For *convex* functions, by Lemma 21, the convex base learner guarantees the following:

$$\textsc{Base-Reg} \leq 5D\sqrt{1 + \bar{V}_{T,i^\star}^{\text{c}}} + \kappa D^2 - \frac{1}{4}\kappa S_{T,i^\star}^{\mathbf{x}} + \mathcal{O}(1)$$
$$\leq 5D\sqrt{1 + 2V_T + 2L^2 S_T^{\mathbf{x}}} + \kappa D^2 - \frac{1}{4}\kappa S_{T,i^\star}^{\mathbf{x}} + \mathcal{O}(1)$$
$$\leq 5D\sqrt{1 + 2V_T} + 10DL^2 S_T^{\mathbf{x}} + \kappa D^2 - \frac{1}{4}\kappa S_{T,i^\star}^{\mathbf{x}} + \mathcal{O}(1). \tag{C.13}$$

Overall Regret Analysis. For λ-*strongly convex* functions, combining Eq. (C.5) and Eq. (C.10) and denoting $C_{11} = 128G^2(1+L^2)$, we obtain

$$
\begin{aligned}
\textsc{Reg}_T \leq\ & \mathcal{O}\left(\frac{1}{\lambda}\log V_T\right) + (64+64L^2)G^2 S_T^{\mathbf{x}} + \left(64G^2 + \gamma^{\text{MID}} - \frac{1}{8}\kappa\right) S_{T,i^\star}^{\mathbf{x}} + \frac{1}{4}\kappa D^2 \\
& - \frac{C_0}{2} S_T^{\text{TOP}} - \gamma^{\text{MID}} \sum_{t=2}^{T}\sum_{j=1}^{M} q_{t,j}^{\text{TOP}} \sum_{i=1}^{N} q_{t,j,i}^{\text{MID}} \|\mathbf{x}_{t,i} - \mathbf{x}_{t-1,i}\|^2 - \gamma^{\text{TOP}} \sum_{t=2}^{T}\sum_{j=1}^{M} q_{t,j}^{\text{TOP}} \|\boldsymbol{q}_{t,j}^{\text{MID}} - \boldsymbol{q}_{t-1,j}^{\text{MID}}\|_1^2 \\
\leq\ & \mathcal{O}\left(\frac{1}{\lambda}\log V_T\right) + \left(64G^2 + \gamma^{\text{MID}} - \frac{1}{8}\kappa\right) S_{T,i^\star}^{\mathbf{x}} + \frac{1}{4}\kappa D^2 \\
& + \left(2D^2 C_{11} - \frac{C_0}{2}\right) S_T^{\text{TOP}} + (C_{11} - \gamma^{\text{MID}}) \sum_{t=2}^{T}\sum_{j=1}^{M} q_{t,j}^{\text{TOP}} \sum_{i=1}^{N} q_{t,j,i}^{\text{MID}} \|\mathbf{x}_{t,i} - \mathbf{x}_{t-1,i}\|^2 \\
& + \left(2D^2 C_{11} - \gamma^{\text{TOP}}\right) \sum_{t=2}^{T}\sum_{j=1}^{M} q_{t,j}^{\text{TOP}} \|\boldsymbol{q}_{t,j}^{\text{MID}} - \boldsymbol{q}_{t-1,j}^{\text{MID}}\|_1^2 \leq \mathcal{O}\left(\frac{1}{\lambda}\log V_T\right),
\end{aligned}
$$

where the second step follows from Lemma 5 and the last step requires $\gamma^{\text{TOP}} \geq 2D^2 C_{11}$, $\gamma^{\text{MID}} \geq C_{11}$, $\kappa \geq 512G^2 + 8\gamma^{\text{MID}}$, and $C_0 \geq 4D^2 C_{11}$.

For α-*exp-concave* functions, combining Eq. (C.6) and Eq. (C.12), we obtain

$$
\begin{aligned}
\textsc{Reg}_T \leq\ & \mathcal{O}\left(\frac{d}{\alpha}\log V_T\right) + \frac{2C_{10}}{\kappa} S_T^{\mathbf{x}} + \left(\frac{16G^4}{\kappa} + \gamma^{\text{MID}} - \frac{\kappa}{4}\right) S_{T,i^\star}^{\mathbf{x}} + \frac{1}{2}\kappa D^2 \\
& - \frac{C_0}{2} S_T^{\text{TOP}} - \gamma^{\text{MID}} \sum_{t=2}^{T}\sum_{j=1}^{M} q_{t,j}^{\text{TOP}} \sum_{i=1}^{N} q_{t,j,i}^{\text{MID}} \|\mathbf{x}_{t,i} - \mathbf{x}_{t-1,i}\|^2 - \gamma^{\text{TOP}} \sum_{t=2}^{T}\sum_{j=1}^{M} q_{t,j}^{\text{TOP}} \|\boldsymbol{q}_{t,j}^{\text{MID}} - \boldsymbol{q}_{t-1,j}^{\text{MID}}\|_1^2 \\
\leq\ & \mathcal{O}\left(\frac{d}{\alpha}\log V_T\right) + \left(\frac{16G^4}{\kappa} + \gamma^{\text{MID}} - \frac{\kappa}{4}\right) S_{T,i^\star}^{\mathbf{x}} + \frac{1}{2}\kappa D^2 \\
& + \left(\frac{8D^2 C_{10}}{\kappa} - \frac{C_0}{2}\right) S_T^{\text{TOP}} + \left(\frac{4C_{10}}{\kappa} - \gamma^{\text{MID}}\right) \sum_{t=2}^{T}\sum_{j=1}^{M} q_{t,j}^{\text{TOP}} \sum_{i=1}^{N} q_{t,j,i}^{\text{MID}} \|\mathbf{x}_{t,i} - \mathbf{x}_{t-1,i}\|^2 \\
& + \left(\frac{8D^2 C_{10}}{\kappa} - \gamma^{\text{TOP}}\right) \sum_{t=2}^{T}\sum_{j=1}^{M} q_{t,j}^{\text{TOP}} \|\boldsymbol{q}_{t,j}^{\text{MID}} - \boldsymbol{q}_{t-1,j}^{\text{MID}}\|_1^2 \leq \mathcal{O}\left(\frac{d}{\alpha}\log V_T\right),
\end{aligned}
$$

where the second step follows from Lemma 5 and the last step requires $\gamma^{\text{TOP}} \geq 8D^2 C_{10}$, $\gamma^{\text{MID}} \geq 4C_{10}$, $\kappa \geq 64G^4 + 4\gamma^{\text{MID}}$, and $C_0 \geq 16D^2 C_{10}$.

For *convex* functions, combining Eq. (C.7) and Eq. (C.13), we obtain

$$
\begin{aligned}
\textsc{Reg}_T \leq\ & \mathcal{O}\left(\sqrt{V_T \log V_T}\right) + \left(10DL^2 + \frac{C_1}{2Z}\right) S_T^{\mathbf{x}} + \left(\frac{64G^2}{Z} + \gamma^{\text{MID}} - \frac{\kappa}{4}\right) S_{T,i^\star}^{\mathbf{x}} + \kappa D^2 \\
& - \frac{C_0}{2} S_T^{\text{TOP}} - \gamma^{\text{MID}} \sum_{t=2}^{T}\sum_{j=1}^{M} q_{t,j}^{\text{TOP}} \sum_{i=1}^{N} q_{t,j,i}^{\text{MID}} \|\mathbf{x}_{t,i} - \mathbf{x}_{t-1,i}\|^2 - \gamma^{\text{TOP}} \sum_{t=2}^{T}\sum_{j=1}^{M} q_{t,j}^{\text{TOP}} \|\boldsymbol{q}_{t,j}^{\text{MID}} - \boldsymbol{q}_{t-1,j}^{\text{MID}}\|_1^2 \\
\leq\ & \mathcal{O}\left(\sqrt{V_T \log V_T}\right) + \left(\frac{64G^2}{Z} + \gamma^{\text{MID}} - \frac{\kappa}{4}\right) S_{T,i^\star}^{\mathbf{x}} + \left(40D^3L^2 + \frac{2D^2 C_1}{Z} - \frac{C_0}{2}\right) S_T^{\text{TOP}}
\end{aligned}
$$

$$+ \kappa D^2 + \left(20DL^2 + \frac{C_1}{Z} - \gamma^{\text{MID}}\right) \sum_{t=2}^{T} \sum_{j=1}^{M} q_{t,j}^{\text{TOP}} \sum_{i=1}^{N} q_{t,j,i}^{\text{MID}} \|\mathbf{x}_{t,i} - \mathbf{x}_{t-1,i}\|^2$$
$$+ \left(40D^3L^2 + \frac{2D^2C_1}{Z} - \gamma^{\text{TOP}}\right) \sum_{t=2}^{T} \sum_{j=1}^{M} q_{t,j}^{\text{TOP}} \|\boldsymbol{q}_{t,j}^{\text{MID}} - \boldsymbol{q}_{t-1,j}^{\text{MID}}\|_1^2 \le \mathcal{O}\left(\sqrt{V_T \log V_T}\right),$$

where the second step follows from Lemma 5 and the last step requires $\gamma^{\text{TOP}} \ge 40D^3L^2 + \frac{2D^2C_1}{Z}$, $\gamma^{\text{MID}} \ge 20DL^2 + \frac{C_1}{Z}$, $\kappa \ge 256G^2 + 4\gamma^{\text{MID}}$, and $C_0 \ge 80D^3L^2 + \frac{4D^2C_1}{Z}$.

At last, we determine the specific values of C_0, γ^{TOP}, and γ^{MID}. These parameters need to satisfy the following requirements:

$$C_0 \ge 1,\ C_0 \ge 8D,\ C_0 \ge 4\gamma^{\text{TOP}},\ C_0 \ge 4D^2C_{11},\ C_0 \ge 16D^2C_{10},\ C_0 \ge 80D^3L^2 + \frac{4D^2C_1}{Z},$$
$$\gamma^{\text{TOP}} \ge 2D^2C_{11},\ \gamma^{\text{TOP}} \ge 8D^2C_{10},\ \gamma^{\text{TOP}} \ge 40D^3L^2 + \frac{2D^2C_1}{Z},\ \gamma^{\text{MID}} \ge C_{11},$$
$$\gamma^{\text{MID}} \ge 4C_{10},\ \text{and}\ \gamma^{\text{MID}} \ge 20DL^2 + \frac{C_1}{Z}.$$

As a result, we set

$$C_0 = \max\left\{1, 8D, 4\gamma^{\text{TOP}}, 4D^2C_{11}, 16D^2C_{10}, 80D^3L^2 + 4D^2C_1\right\},$$
$$\gamma^{\text{TOP}} = \max\left\{2D^2C_{11}, 8D^2C_{10}, 40D^3L^2 + 2D^2C_1\right\}, \gamma^{\text{MID}} = \max\left\{C_{11}, 4C_{10}, 20DL^2 + C_1\right\},$$

where $Z = \max\{GD + \gamma^{\text{MID}}D^2, 1 + \gamma^{\text{MID}}D^2 + 2\gamma^{\text{TOP}}\}$, $C_1 = 128(D^2L^2 + G^2)$, $C_{10} = 4L^2 + 32D^2G^2L^2 + 8G^4$, and $C_{11} = 128G^2(1 + L^2)$.

■

C.4 Proof of Theorem 4

Proof Recall that, for simplicity, we denote by $\mathbf{g}_t \triangleq \nabla f_t(\mathbf{x}_t)$. As shown in Section 5.3, the empirical gradient variation can bounded as

$$\bar{V}_T \le 3\sum_{t=2}^{T} \|\nabla f_t(\mathbf{x}_t) - \nabla f_t(\mathbf{x}^\star)\|^2 + 3\sum_{t=2}^{T} \|\nabla f_t(\mathbf{x}^\star) - \nabla f_{t-1}(\mathbf{x}^\star)\|^2$$
$$+ 3\sum_{t=2}^{T} \|\nabla f_{t-1}(\mathbf{x}^\star) - \nabla f_{t-1}(\mathbf{x}_{t-1})\|^2 \le 6L\sum_{t=2}^{T} \mathcal{D}_{f_t}(\mathbf{x}^\star, \mathbf{x}_t) + 3V_T + 6L\sum_{t=2}^{T} \mathcal{D}_{f_{t-1}}(\mathbf{x}^\star, \mathbf{x}_{t-1})$$
$$\le 3V_T + 12L\sum_{t=1}^{T} \mathcal{D}_{f_t}(\mathbf{x}^\star, \mathbf{x}_t). \tag{C.14}$$

Regret Decomposition. For λ-*strongly convex* functions, we decompose the regret as

$$\begin{aligned}\textsc{Reg}_T \leq{} & \underbrace{\sum_{t=1}^T \langle \mathbf{g}_t, \mathbf{x}_t - \mathbf{x}_{t,i^\star}\rangle - \frac{\lambda_{i^\star}}{4}\sum_{t=1}^T \|\mathbf{x}_t - \mathbf{x}_{t,i^\star}\|^2}_{\textsc{Meta-Reg}} && (\text{by } \lambda_{i^\star} \leq \lambda \leq 2\lambda_{i^\star}) \\ & + \underbrace{\sum_{t=1}^T h^{\text{sc}}_{t,i^\star}(\mathbf{x}_{t,i^\star}) - \sum_{t=1}^T h^{\text{sc}}_{t,i^\star}(\mathbf{x}^\star)}_{\textsc{Base-Reg}} - \frac{1}{2}\sum_{t=1}^T \mathcal{D}_{f_t}(\mathbf{x}^\star, \mathbf{x}_t), && \text{(C.15)}\end{aligned}$$

due to the definition of the surrogate $h^{\text{sc}}_{t,i}(\mathbf{x}) \triangleq \langle \mathbf{g}_t, \mathbf{x}\rangle + \frac{\lambda_i}{4}\|\mathbf{x} - \mathbf{x}_t\|^2$, where $\lambda_i \in \mathcal{H}$ in (2.5). For α-*exp-concave* functions, we decompose the regret as

$$\begin{aligned}\textsc{Reg}_T ={} & \sum_{t=1}^T \langle \mathbf{g}_t, \mathbf{x}_t - \mathbf{x}^\star\rangle - \frac{1}{2}\sum_{t=1}^T \mathcal{D}_{f_t}(\mathbf{x}^\star, \mathbf{x}_t) - \frac{1}{2}\sum_{t=1}^T \mathcal{D}_{f_t}(\mathbf{x}^\star, \mathbf{x}_t) && (\text{by (5.5)}) \\ \leq{} & \sum_{t=1}^T \langle \mathbf{g}_t, \mathbf{x}_t - \mathbf{x}^\star\rangle - \frac{\alpha}{4}\sum_{t=1}^T \langle \mathbf{g}_t, \mathbf{x}_t - \mathbf{x}^\star\rangle^2 - \frac{1}{2}\sum_{t=1}^T \mathcal{D}_{f_t}(\mathbf{x}^\star, \mathbf{x}_t) \\ \leq{} & \underbrace{\sum_{t=1}^T \langle \mathbf{g}_t, \mathbf{x}_t - \mathbf{x}_{t,i^\star}\rangle - \frac{\alpha_{i^\star}}{4}\sum_{t=1}^T \langle \mathbf{g}_t, \mathbf{x}_t - \mathbf{x}_{t,i^\star}\rangle^2}_{\textsc{Meta-Reg}} && (\text{by } \alpha_{i^\star} \leq \alpha \leq 2\alpha_{i^\star}) \\ & + \underbrace{\sum_{t=1}^T h^{\exp}_{t,i^\star}(\mathbf{x}_{t,i^\star}) - \sum_{t=1}^T h^{\exp}_{t,i^\star}(\mathbf{x}^\star)}_{\textsc{Base-Reg}} - \frac{1}{2}\sum_{t=1}^T \mathcal{D}_{f_t}(\mathbf{x}^\star, \mathbf{x}_t), && \text{(C.16)}\end{aligned}$$

where the second step is due to the definitions of exp-concavity and Bregman divergence and the last step is due to the definition of the surrogate $h^{\exp}_{t,i}(\mathbf{x}) \triangleq \langle \mathbf{g}_t, \mathbf{x}\rangle + \frac{\alpha_i}{4}\langle \nabla f_t(\mathbf{x}_t), \mathbf{x} - \mathbf{x}_t\rangle^2$, where $\alpha_i \in \mathcal{H}$, defined in (2.5). For *convex* functions, by defining $h^{\text{c}}_{t,i}(\mathbf{x}) \triangleq \langle \mathbf{g}_t, \mathbf{x}\rangle$, we have

$$\begin{aligned}\textsc{Reg}_T ={} & \sum_{t=1}^T \langle \mathbf{g}_t, \mathbf{x}_t - \mathbf{x}^\star\rangle - \sum_{t=1}^T \mathcal{D}_{f_t}(\mathbf{x}^\star, \mathbf{x}_t) && (\text{by (5.5)}) \\ ={} & \underbrace{\sum_{t=1}^T \langle \mathbf{g}_t, \mathbf{x}_t - \mathbf{x}_{t,i^\star}\rangle}_{\textsc{Meta-Reg}} + \underbrace{\sum_{t=1}^T h^{\text{c}}_{t,i^\star}(\mathbf{x}_{t,i^\star}) - \sum_{t=1}^T h^{\text{c}}_{t,i^\star}(\mathbf{x}^\star)}_{\textsc{Base-Reg}} - \sum_{t=1}^T \mathcal{D}_{f_t}(\mathbf{x}^\star, \mathbf{x}_t). && \text{(C.17)}\end{aligned}$$

Meta Regret Analysis. For *strongly convex* functions and *exp-concave* functions, we adopt a similar meta regret analysis as used in Appendix B.3.

For *convex* functions, similar to Eq. (B.10), we obtain

$$\begin{aligned}\textsc{Meta-Reg} \leq{} & C_0\sqrt{4G^2D^2 + \sum_{t=1}^T \langle \mathbf{g}_t - \mathbf{g}_{t-1}, \mathbf{x}_t - \mathbf{x}_{t,i^\star}\rangle^2} + 2GDC_2 \\ \leq{} & C_0\sqrt{1 + D^2\bar{V}_T} + 2GDC_2 \leq C_0\sqrt{4G^2D^2 + 3D^2V_T + 12LD^2\sum_{t=1}^T \mathcal{D}_{f_t}(\mathbf{x}^\star, \mathbf{x}_t)} + 2GDC_2\end{aligned}$$

$$\leq \mathcal{O}(\sqrt{V_T}) + C_0\sqrt{12LD^2\sum_{t=1}^T \mathcal{D}_{f_t}(\mathbf{x}^\star, \mathbf{x}_t)} \leq \mathcal{O}(\sqrt{V_T}) + \mathcal{O}(C_{12}) + \frac{C_0}{2C_{12}}\sum_{t=1}^T \mathcal{D}_{f_t}(\mathbf{x}^\star, \mathbf{x}_t), \tag{C.18}$$

where the second step adopts Assumption 1 and the third step is by Eq. (B.3). C_{12} is used to ensure the positive Bregman divergence term to be canceled and will be specified finally.

Base Regret Analysis. Following the analysis structure of Appendix C.3, we first provide different decompositions of the empirical gradient variation defined on surrogates for strongly convex, exp-concave, and convex functions, respectively, and then analyze the base regret in the corresponding cases.

For λ-*strongly convex* functions, we bound the empirical gradient variation on surrogates, i.e., $\bar{V}^{\text{sc}}_{T,i^\star} \triangleq \sum_{t=2}^T \|\nabla h^{\text{sc}}_{t,i^\star}(\mathbf{x}_{t,i^\star}) - \nabla h^{\text{sc}}_{t-1,i^\star}(\mathbf{x}_{t-1,i^\star})\|^2$, by

$$\begin{aligned}
\bar{V}^{\text{sc}}_{T,i^\star} &= \sum_{t=2}^T \left\|\mathbf{g}_t + \frac{\lambda_{i^\star}}{2}(\mathbf{x}_{t,i^\star} - \mathbf{x}_t) - \mathbf{g}_{t-1} - \frac{\lambda_{i^\star}}{2}(\mathbf{x}_{t-1,i^\star} - \mathbf{x}_{t-1})\right\|^2 \\
&\leq 3\bar{V}_T + 3\sum_{t=2}^T \left\|\frac{\lambda_{i^\star}}{2}(\mathbf{x}_{t,i^\star} - \mathbf{x}_t)\right\|^2 + 3\sum_{t=2}^T \left\|\frac{\lambda_{i^\star}}{2}(\mathbf{x}_{t-1,i^\star} - \mathbf{x}_{t-1})\right\|^2 && \text{(C.19)} \\
&\leq 9V_T + 36L\sum_{t=1}^T \mathcal{D}_{f_t}(\mathbf{x}^\star, \mathbf{x}_t) + 2\lambda_{i^\star}^2 \sum_{t=2}^T \|\mathbf{x}_{t,i^\star} - \mathbf{x}_t\|^2, && \text{(by (C.14))}
\end{aligned}$$

where the first step is due to the property of the surrogate: $\nabla h^{\text{sc}}_{t,i}(\mathbf{x}_{t,i}) = \mathbf{g}_t + \frac{\lambda_i}{2}(\mathbf{x}_{t,i} - \mathbf{x}_t)$, and the second step is due to the Cauchy-Schwarz inequality. For α-*exp-concave* functions, we control the $\bar{V}^{\text{exp}}_{T,i^\star} \triangleq \sum_{t=2}^T \|\nabla h^{\text{exp}}_{t,i^\star}(\mathbf{x}_{t,i^\star}) - \nabla h^{\text{exp}}_{t-1,i^\star}(\mathbf{x}_{t-1,i^\star})\|^2$ as

$$\begin{aligned}
\bar{V}^{\text{exp}}_{T,i^\star} &= \sum_{t=2}^T \left\|\mathbf{g}_t + \frac{\alpha_{i^\star}}{2}\mathbf{g}_t\langle \mathbf{g}_t, \mathbf{x}_t - \mathbf{x}_{t,i^\star}\rangle - \mathbf{g}_{t-1} - \frac{\alpha_{i^\star}}{2}\mathbf{g}_{t-1}\langle \mathbf{g}_{t-1}, \mathbf{x}_{t-1} - \mathbf{x}_{t-1,i^\star}\rangle\right\|^2 \\
&\leq 3\bar{V}_T + 3\sum_{t=2}^T \left\|\frac{\alpha_{i^\star}}{2}\mathbf{g}_t\langle \mathbf{g}_t, \mathbf{x}_t - \mathbf{x}_{t,i^\star}\rangle\right\|^2 + 3\sum_{t=2}^T \left\|\frac{\alpha_{i^\star}}{2}\mathbf{g}_{t-1}\langle \mathbf{g}_{t-1}, \mathbf{x}_{t-1} - \mathbf{x}_{t-1,i^\star}\rangle\right\|^2 \\
&\leq 3\bar{V}_T + 6\sum_{t=1}^T \left\|\frac{\alpha_{i^\star}}{2}\mathbf{g}_t\langle \mathbf{g}_t, \mathbf{x}_t - \mathbf{x}_{t,i^\star}\rangle\right\|^2 && \text{(C.20)} \\
&\overset{\text{(C.14)}}{\leq} 9V_T + 36L\sum_{t=1}^T \mathcal{D}_{f_t}(\mathbf{x}^\star, \mathbf{x}_t) + 2\alpha_{i^\star}^2 G^2 \sum_{t=1}^T \langle \mathbf{g}_t, \mathbf{x}_t - \mathbf{x}_{t,i^\star}\rangle^2, && \text{(by Assumption 2)}
\end{aligned}$$

where the first step is due to the property of the surrogate function: $\nabla h^{\text{exp}}_{t,i}(\mathbf{x}_{t,i}) = \mathbf{g}_t + \frac{\alpha_i}{2}\mathbf{g}_t\langle \mathbf{g}_t, \mathbf{x}_t - \mathbf{x}_{t,i}\rangle$, the second step is by the Cauchy-Schwarz inequality. For *convex* functions, the empirical gradient variation $\bar{V}^{\text{c}}_{T,i^\star} \triangleq \sum_{t=2}^T \|\nabla f_t(\mathbf{x}_t) - \nabla f_{t-1}(\mathbf{x}_{t-1})\|^2$ can be bounded by $\bar{V}^{\text{c}}_{T,i^\star} \leq 2V_T + 2L^2 S^{\mathbf{x}}_T$. To conclude, for different curvature types, we provide correspondingly different analysis of the empirical gradient variation on surrogates:

$$\bar{V}_{T,i^\star}^{\{\mathrm{sc},\exp,\mathrm{c}\}} \le \begin{cases} 9V_T + 36L\sum_{t=1}^T \mathcal{D}_{f_t}(\mathbf{x}^\star, \mathbf{x}_t) + 2\lambda_{i^\star}^2 \sum_{t=2}^T \|\mathbf{x}_{t,i^\star} - \mathbf{x}_t\|^2, & (\lambda\text{-strongly convex}), \\ 9V_T + 36L\sum_{t=1}^T \mathcal{D}_{f_t}(\mathbf{x}^\star, \mathbf{x}_t) + 2\alpha_{i^\star}^2 G^2 \sum_{t=1}^T \langle \mathbf{g}_t, \mathbf{x}_t - \mathbf{x}_{t,i^\star}\rangle^2, & (\alpha\text{-exp-concave}), \\ 3V_T + 12L\sum_{t=1}^T \mathcal{D}_{f_t}(\mathbf{x}^\star, \mathbf{x}_t), & (\text{convex}). \end{cases} \tag{C.21}$$

In the following, we analyze the base regret for different curvature types. For λ-*strongly convex* functions, when using the update rule (C.2), according to Lemma 19, the base regret can be bounded as

$$\begin{aligned} &\textsc{Base-Reg} \le \frac{16G^2}{\lambda_{i^\star}} \log\left(1 + \lambda_{i^\star}\bar{V}_{T,i^\star}^{\mathrm{sc}}\right) + \mathcal{O}(1) \\ &\le \frac{16G^2}{\lambda_{i^\star}} \log\left(1 + 9\lambda_{i^\star}V_T + 36L\lambda_{i^\star}\sum_{t=1}^T \mathcal{D}_{f_t}(\mathbf{x}^\star, \mathbf{x}_t) + 2\lambda_{i^\star}^3 \sum_{t=1}^T \|\mathbf{x}_{t,i^\star} - \mathbf{x}_t\|^2\right) \\ &\le \mathcal{O}\left(\frac{1}{\lambda}\log(C_{13}V_T)\right) + \frac{16G^2}{C_{13}\lambda_{i^\star}}\left(36L\lambda_{i^\star}\sum_{t=1}^T \mathcal{D}_{f_t}(\mathbf{x}^\star, \mathbf{x}_t) + 2\lambda_{i^\star}^3\sum_{t=1}^T \|\mathbf{x}_{t,i^\star} - \mathbf{x}_t\|^2\right) \\ &\le \mathcal{O}\left(\frac{1}{\lambda}\log V_T\right) + \frac{576G^2L}{C_{13}}\sum_{t=1}^T \mathcal{D}_{f_t}(\mathbf{x}^\star, \mathbf{x}_t) + \frac{32G^2}{C_{13}}\sum_{t=1}^T \|\mathbf{x}_{t,i^\star} - \mathbf{x}_t\|^2 + \mathcal{O}(\log C_{13}), \end{aligned} \tag{C.22}$$

where the third step requires $C_{13} \ge 1$ by Lemma 15 and uses the property of the best base learner, i.e., $\lambda_{i^\star} \le \lambda \le 2\lambda_{i^\star}$. The last step is due to $\lambda_i \le 1$. For *exp-concave* functions, when using the update rule (C.1), by Lemma 20, the base regret can be bounded as

$$\begin{aligned} &\textsc{Base-Reg} \le \frac{16d}{\alpha_{i^\star}} \log\left(1 + \frac{\alpha_{i^\star}}{8d}\bar{V}_{T,i^\star}^{\exp}\right) + \mathcal{O}(1) \\ &\le \frac{16d}{\alpha_{i^\star}} \log\left(1 + \frac{9\alpha_{i^\star}}{8d}V_T + \frac{9\alpha_{i^\star}L}{2d}\sum_{t=1}^T \mathcal{D}_{f_t}(\mathbf{x}^\star, \mathbf{x}_t) + \frac{\alpha_{i^\star}^3G^2}{4d}\sum_{t=1}^T \langle \mathbf{g}_t, \mathbf{x}_t - \mathbf{x}_{t,i^\star}\rangle^2\right) \\ &\le \mathcal{O}\left(\frac{d}{\alpha}\log(C_{14}V_T)\right) + \frac{16d}{C_{14}\alpha_{i^\star}}\left(\frac{9\alpha_{i^\star}L}{2d}\sum_{t=1}^T \mathcal{D}_{f_t}(\mathbf{x}^\star, \mathbf{x}_t) + \frac{\alpha_{i^\star}^3G^2}{4d}\sum_{t=1}^T \langle \mathbf{g}_t, \mathbf{x}_t - \mathbf{x}_{t,i^\star}\rangle^2\right) \\ &\le \mathcal{O}\left(\frac{d}{\alpha}\log V_T\right) + \frac{72L}{C_{14}}\sum_{t=1}^T \mathcal{D}_{f_t}(\mathbf{x}^\star, \mathbf{x}_t) + \frac{4G^2}{C_{14}}\sum_{t=1}^T \langle \mathbf{g}_t, \mathbf{x}_t - \mathbf{x}_{t,i^\star}\rangle^2 + \mathcal{O}(\log C_{14}). \end{aligned} \tag{C.23}$$

The third step requires $C_{14} \ge 1$ by Lemma 15 and uses the property of the best base learner, i.e., $\alpha_{i^\star} \le \alpha \le 2\alpha_{i^\star}$. The last step is because of $\alpha_i \le 1$. For *convex* functions, when using the update rule (C.3), by Lemma 21, we obtain

$$\begin{aligned} &\textsc{Base-Reg} \le 5D\sqrt{1 + \bar{V}_{T,i^\star}^{\mathrm{c}}} + \mathcal{O}(1) \\ &= 5D\sqrt{1 + \bar{V}_T} + \mathcal{O}(1) \le \mathcal{O}(\sqrt{V_T}) + 5D\sqrt{12L\sum_{t=1}^T \mathcal{D}_{f_t}(\mathbf{x}^\star, \mathbf{x}_t)} \qquad (\text{by (C.14)}) \\ &\le \mathcal{O}(\sqrt{V_T}) + \mathcal{O}(C_{15}) + \frac{5DL}{2C_{15}}\sum_{t=1}^T \mathcal{D}_{f_t}(\mathbf{x}^\star, \mathbf{x}_t), \end{aligned} \tag{C.24}$$

where the second step is due to the property of the surrogate function: $\nabla h^{c}_{t,i}(\mathbf{x}_{t,i}) = \mathbf{g}_t$, and the last step uses AM-GM inequality (Lemma 18). C_{15} is a constant to be specified.

Overall Regret Analysis. For λ-*strongly convex* functions, plugging Eq. (B.8) and Eq. (C.22) into Eq. (C.15), we obtain

$$\textsc{Reg}_T \le \mathcal{O}\left(\frac{1}{\lambda}\log V_T\right) + \mathcal{O}(C_3 + \log C_{13}) + \left(\frac{576G^2L}{C_{13}} - \frac{1}{2}\right)\sum_{t=1}^{T}\mathcal{D}_{f_t}(\mathbf{x}^\star, \mathbf{x}_t)$$
$$+ \left(\frac{C_0D^2}{2C_3} + \frac{32G^2}{C_{13}} - \frac{\lambda}{4}\right)\sum_{t=1}^{T}\|\mathbf{x}_t - \mathbf{x}_{t,i^\star}\|^2 \le \mathcal{O}\left(\frac{1}{\lambda}\log V_T\right),$$

by choosing $C_3 = 4C_0D^2/\lambda$ and $C_{13} = \max\{1, 256G^2/\lambda, 1152G^2L\}$. For α-*exp-concave* functions, plugging Eq. (B.9) and Eq. (C.23) into Eq. (C.16), we obtain

$$\textsc{Reg}_T \le \mathcal{O}\left(\frac{d}{\alpha}\log V_T\right) + \mathcal{O}(C_4 + \log C_{14}) + \left(\frac{72L}{C_{14}} - \frac{1}{2}\right)\sum_{t=1}^{T}\mathcal{D}_{f_t}(\mathbf{x}^\star, \mathbf{x}_t)$$
$$+ \left(\frac{C_0}{2C_4} + \frac{4G^2}{C_{14}} - \frac{\alpha}{4}\right)\sum_{t=1}^{T}\langle \mathbf{g}_t, \mathbf{x}_t - \mathbf{x}_{t,i^\star}\rangle^2 \le \mathcal{O}\left(\frac{d}{\alpha}\log V_T\right),$$

by choosing $C_4 = 4C_0/\alpha$ and $C_{14} = \max\{1, 144L, 32G^2/\alpha\}$. For *convex* functions, plugging Eq. (C.18) and Eq. (C.24) into Eq. (C.17), we obtain

$$\textsc{Reg}_T \le \mathcal{O}(\sqrt{V_T}) + \mathcal{O}(C_{12} + C_{15}) + \left(\frac{5DL}{2C_{15}} + \frac{C_0}{2C_{12}} - 1\right)\sum_{t=1}^{T}\mathcal{D}_{f_t}(\mathbf{x}^\star, \mathbf{x}_t) \le \mathcal{O}(\sqrt{V_T}),$$

by choosing $C_{12} = C_0$ and $C_{15} = 5DL$.

Note that the constants $C_3, C_4, C_{12}, C_{13}, C_{14}, C_{15}$ only exist in analysis and hence our choices of them are feasible. ■

Appendix D. Omitted Proofs for Section 6

In this section, we present the formal proofs supporting the theoretical implications of our methods, specifically regarding the small-loss and gradient-variance bounds discussed in Section 6.1. We also provide complete proofs for the applications of our results to the SEA model (Section 6.2) and to online games (Section 6.3), including detailed proofs of Corollary 1, Corollary 2, Theorem 5, Theorem 6, and Theorem 7. Finally, we provide the proof of the extension to anytime variant (Section 6.4).

D.1 Proof of Corollary 1

We prove the small-loss regret guarantees of UniGrad++.Correct in Appendix D.1.1 and the gradient-variance regret guarantees of UniGrad++.Correct in Appendix D.1.2.

D.1.1 Small-Loss Regret

Proof For simplicity, we define $F_T^{\mathbf{x}} \triangleq \sum_{t=1}^{T} f_t(\mathbf{x}_t) - \sum_{t=1}^{T}\min_{\mathbf{x}\in\mathcal{X}_+} f_t(\mathbf{x})$. We adopt the same regret decomposition strategy as utilized in Appendix C.3.

Meta Regret Analysis. Recall that the normalization factor $Z = \max\{GD+\gamma^{\text{MID}} D^2, 1+\gamma^{\text{MID}} D^2 + 2\gamma^{\text{TOP}}\}$. For *strongly convex* and *exp-concave* functions, we follow the same meta regret analysis as used in Appendix C.3.

For *convex* functions, we give a different analysis for $V_\star = \sum_{t=2}^T (\ell^{\text{MID}}_{t,j^\star,i^\star} - m^{\text{MID}}_{t,j^\star,i^\star})^2$. From Lemma 3, it holds that

$$\begin{aligned} V_\star &\le \frac{2D^2}{Z^2}\sum_{t=1}^T \|\mathbf{g}_t - \mathbf{g}_{t-1}\|^2 + \frac{2G^2}{Z^2}\sum_{t=1}^T \|\mathbf{x}_{t,i^\star} - \mathbf{x}_{t-1,i^\star} + \mathbf{x}_{t-1} - \mathbf{x}_t\|^2 \\ &\le \frac{8D^2}{Z^2}\sum_{t=1}^T \|\mathbf{g}_t\|^2 + \frac{4G^2}{Z^2} S^{\mathbf{x}}_{T,i^\star} + \frac{4G^2}{Z^2} S^{\mathbf{x}}_T \le \frac{32D^2L}{Z^2} F^{\mathbf{x}}_T + \frac{4G^2}{Z^2} S^{\mathbf{x}}_{T,i^\star} + \frac{4G^2}{Z^2} S^{\mathbf{x}}_T, \end{aligned} \tag{D.1}$$

where the last step is by the self-bounding property of $\|\nabla f(\mathbf{x})\|_2^2 \le 4L(f(\mathbf{x}) - \min_{\mathbf{x}\in\mathcal{X}_+} f(\mathbf{x}))$ for any $\mathbf{x} \in \mathcal{X}$. Thus, the meta regret is bounded as

$$\begin{aligned} &\textsc{Meta-Reg} \le \frac{Z}{\varepsilon^{\text{TOP}}_{j^\star}} \log \frac{N}{3C_0^2(\varepsilon^{\text{TOP}}_{j^\star})^2} + 32Z\varepsilon^{\text{TOP}}_{j^\star} V_\star + \gamma^{\text{MID}} S^{\mathbf{x}}_{T,i^\star} - \frac{C_0}{2} S^{\text{TOP}}_T \\ &\quad - \frac{C_0}{2} S^{\text{TOP}}_T - \gamma^{\text{MID}} \sum_{t=2}^T \sum_{j=1}^M q^{\text{TOP}}_{t,j} \sum_{i=1}^N q^{\text{MID}}_{t,j,i} \|\mathbf{x}_{t,i} - \mathbf{x}_{t-1,i}\|^2 - \gamma^{\text{TOP}} \sum_{t=2}^T \sum_{j=1}^M q^{\text{TOP}}_{t,j} \|\boldsymbol{q}^{\text{MID}}_{t,j} - \boldsymbol{q}^{\text{MID}}_{t-1,j}\|_1^2 \\ &\le \frac{Z}{\varepsilon^{\text{TOP}}_{j^\star}} \log \frac{N}{3C_0^2(\varepsilon^{\text{TOP}}_{j^\star})^2} + \frac{1024D^2L}{Z}\varepsilon^{\text{TOP}}_{j^\star} F^{\mathbf{x}}_T + \left(\frac{64G^2}{Z} + \gamma^{\text{MID}}\right) S^{\mathbf{x}}_{T,i^\star} + \frac{64G^2}{Z} S^{\mathbf{x}}_T \\ &\quad - \frac{C_0}{2} S^{\text{TOP}}_T - \gamma^{\text{MID}} \sum_{t=2}^T \sum_{j=1}^M q^{\text{TOP}}_{t,j} \sum_{i=1}^N q^{\text{MID}}_{t,j,i} \|\mathbf{x}_{t,i} - \mathbf{x}_{t-1,i}\|^2 - \gamma^{\text{TOP}} \sum_{t=2}^T \sum_{j=1}^M q^{\text{TOP}}_{t,j} \|\boldsymbol{q}^{\text{MID}}_{t,j} - \boldsymbol{q}^{\text{MID}}_{t-1,j}\|_1^2 \\ &\le \frac{Z}{\varepsilon^{\text{TOP}}_{j^\star}} \log \frac{N}{3C_0^2(\varepsilon^{\text{TOP}}_{j^\star})^2} + \frac{1024D^2L}{Z}\varepsilon^{\text{TOP}}_{j^\star} F^{\mathbf{x}}_T + \left(\frac{64G^2}{Z} + \gamma^{\text{MID}}\right) S^{\mathbf{x}}_{T,i^\star} \\ &\quad + \left(\frac{256D^2G^2}{Z} - \frac{C_0}{2}\right) S^{\text{TOP}}_T + \left(\frac{128G^2}{Z} - \gamma^{\text{MID}}\right) \sum_{t=2}^T \sum_{j=1}^M q^{\text{TOP}}_{t,j} \sum_{i=1}^N q^{\text{MID}}_{t,j,i} \|\mathbf{x}_{t,i} - \mathbf{x}_{t-1,i}\|^2 \\ &\quad + \left(\frac{256D^2G^2}{Z} - \gamma^{\text{TOP}}\right) \sum_{t=2}^T \sum_{j=1}^M q^{\text{TOP}}_{t,j} \|\boldsymbol{q}^{\text{MID}}_{t,j} - \boldsymbol{q}^{\text{MID}}_{t-1,j}\|_1^2 \\ &\le \frac{Z}{\varepsilon^{\text{TOP}}_{j^\star}} \log \frac{N}{3C_0^2(\varepsilon^{\text{TOP}}_{j^\star})^2} + \frac{1024D^2L}{Z}\varepsilon^{\text{TOP}}_{j^\star} F^{\mathbf{x}}_T + \left(\frac{64G^2}{Z} + \gamma^{\text{MID}}\right) S^{\mathbf{x}}_{T,i^\star}, \end{aligned} \tag{D.2}$$

where the second step is by Lemma 12 and requiring $C_0 \ge 8D$, the third step is by Lemma 5, and the final step requires $\gamma^{\text{TOP}} \ge \frac{256D^2G^2}{Z}, \gamma^{\text{MID}} \ge \frac{128G^2}{Z}$, and $C_0 \ge \frac{512D^2G^2}{Z}$.

Base Regret Analysis. We first provide different decompositions of the empirical gradient variation defined on surrogates for strongly convex, exp-concave, and convex functions, respectively, and then analyze the base regret in the corresponding cases. For λ-*strongly* *convex* functions, $\bar{V}_{T,i}^{\text{sc}} \triangleq \sum_{t=2}^T \|\nabla h_{t,i}^{\text{sc}}(\mathbf{x}_{t,i^\star}) - \nabla h_{t-1,i}^{\text{sc}}(\mathbf{x}_{t-1,i^\star})\|^2$ can be bounded by

$$\begin{aligned}
\bar{V}_{T,i^\star}^{\text{sc}} &= \sum_{t=2}^T \|(\mathbf{g}_t + \lambda_{i^\star}(\mathbf{x}_{t,i^\star} - \mathbf{x}_t)) - (\mathbf{g}_{t-1} + \lambda_{i^\star}(\mathbf{x}_{t-1,i^\star} - \mathbf{x}_{t-1}))\|^2 \\
&\le 2\sum_{t=2}^T \|\mathbf{g}_t - \mathbf{g}_{t-1}\|^2 + 2\lambda_{i^\star}^2 \sum_{t=2}^T \|(\mathbf{x}_{t,i^\star} - \mathbf{x}_t) - (\mathbf{x}_{t-1,i^\star} - \mathbf{x}_{t-1})\|^2 \\
&\le 32LF_T^{\mathbf{x}} + 4S_T^{\mathbf{x}} + 4S_{T,i^\star}^{\mathbf{x}}, \qquad \text{(by } \lambda \in [1/T, 1])
\end{aligned}$$

where the first step uses the definition of $\nabla h_{t,i}^{\text{sc}}(\mathbf{x})$. For α-*exp-concave* functions, $\bar{V}_{T,i^\star}^{\exp} \triangleq \sum_{t=2}^T \|\nabla h_{t,i^\star}^{\exp}(\mathbf{x}_{t,i^\star}) - \nabla h_{t-1,i^\star}^{\exp}(\mathbf{x}_{t-1,i^\star})\|^2$ can be bounded by

$$\begin{aligned}
\bar{V}_{T,i^\star}^{\exp} &= \sum_{t=2}^T \|(\mathbf{g}_t + \alpha_{i^\star}\mathbf{g}_t\mathbf{g}_t^\top(\mathbf{x}_{t,i^\star} - \mathbf{x}_t)) - (\mathbf{g}_{t-1} + \alpha_{i^\star}\mathbf{g}_{t-1}\mathbf{g}_{t-1}^\top(\mathbf{x}_{t-1,i^\star} - \mathbf{x}_{t-1}))\|^2 \\
&\le 2\sum_{t=2}^T \|\mathbf{g}_t - \mathbf{g}_{t-1}\|^2 + 2\alpha_{i^\star}^2 \sum_{t=2}^T \|\mathbf{g}_t\mathbf{g}_t^\top(\mathbf{x}_{t,i^\star} - \mathbf{x}_t) - \mathbf{g}_{t-1}\mathbf{g}_{t-1}^\top(\mathbf{x}_{t-1,i^\star} - \mathbf{x}_{t-1})\|^2 \\
&\le 32LF_T^{\mathbf{x}} + 4D^2 \sum_{t=2}^T \|\mathbf{g}_t\mathbf{g}_t^\top - \mathbf{g}_{t-1}\mathbf{g}_{t-1}^\top\|^2 + 4G^4 \sum_{t=2}^T \|(\mathbf{x}_{t,i^\star} - \mathbf{x}_t) - (\mathbf{x}_{t-1,i^\star} - \mathbf{x}_{t-1})\|^2 \\
&\qquad \text{(by } \alpha \in [1/T, 1]) \\
&\le C_{29}F_T^{\mathbf{x}} + 8G^4S_T^{\mathbf{x}} + 8G^4S_{T,i^\star}^{\mathbf{x}},
\end{aligned}$$

where the first step uses the definition of $\nabla h_{t,i}^{\exp}(\mathbf{x}) = \mathbf{g}_t + \alpha_i\mathbf{g}_t\mathbf{g}_t^\top(\mathbf{x} - \mathbf{x}_t)$ and the last step holds by setting $C_{29} = 32L + 256D^2G^2L$.

For *convex* functions, $\bar{V}_{T,i^\star}^{\text{c}}$ can be bounded by $\bar{V}_{T,i^\star}^{\text{c}} \triangleq \sum_{t=2}^T \|\mathbf{g}_t - \mathbf{g}_{t-1}\|^2 \le 16LF_T^{\mathbf{x}}$. To conclude, for different curvature types, we provide correspondingly different analysis of the empirical gradient variation on surrogates:

$$\bar{V}_{T,i^\star}^{\{\text{sc,exp,c}\}} \le \begin{cases} 32LF_T^{\mathbf{x}} + 4S_T^{\mathbf{x}} + 4S_{T,i^\star}^{\mathbf{x}}, & \text{when } \{f_t\}_{t=1}^T \text{ are } \lambda\text{-strongly convex}, \\ C_{29}F_T^{\mathbf{x}} + 8G^4S_T^{\mathbf{x}} + 8G^4S_{T,i^\star}^{\mathbf{x}}, & \text{when } \{f_t\}_{t=1}^T \text{ are } \alpha\text{-exp-concave}, \\ 16LF_T^{\mathbf{x}}, & \text{when } \{f_t\}_{t=1}^T \text{ are convex.} \end{cases} \tag{D.3}$$

In the following, we analyze the base regret for different curvature types. For λ-*strongly convex* functions, by Lemma 19, the $i^\star$-th base learner guarantees the following:

$$\begin{aligned}
&\textsc{Base-Reg} \le \frac{16G^2}{\lambda_{i^\star}} \log(1 + 32\lambda_{i^\star}LF_T^{\mathbf{x}} + 4\lambda_{i^\star}S_T^{\mathbf{x}} + 4\lambda_{i^\star}S_{T,i^\star}^{\mathbf{x}}) + \frac{1}{4}\kappa D^2 - \frac{1}{8}\kappa S_{T,i^\star}^{\mathbf{x}} + \mathcal{O}(1) \\
&\le \frac{32G^2}{\lambda} \log(1 + 32\lambda LF_T^{\mathbf{x}}) + 64G^2S_T^{\mathbf{x}} + 64G^2S_{T,i^\star}^{\mathbf{x}} + \frac{1}{4}\kappa D^2 - \frac{1}{8}\kappa S_{T,i^\star}^{\mathbf{x}} + \mathcal{O}(1),
\end{aligned} \tag{D.4}$$

where the last step is due to $\log(1+x) \le x$ for $x \ge 0$. For α-*exp-concave* functions, by Lemma 20, the $i^\star$-th base learner guarantees the following:

$$\begin{aligned}
&\textsc{Base-Reg} \le \frac{16d}{\alpha_{i^\star}} \log\left(1 + \frac{\alpha_{i^\star} C_{29}}{8\kappa d} F_T^{\mathbf{x}} + \frac{\alpha_{i^\star} G^4}{\kappa d} S_T^{\mathbf{x}} + \frac{\alpha_{i^\star} G^4}{\kappa d} S_{T,i^\star}^{\mathbf{x}}\right) + \frac{1}{2}\kappa D^2 - \frac{1}{4}\kappa S_{T,i^\star}^{\mathbf{x}} + \mathcal{O}(1) \\
&\le \frac{32d}{\alpha} \log\left(1 + \frac{\alpha C_{29}}{8\kappa d} F_T^{\mathbf{x}}\right) + \frac{16G^4}{\kappa} S_T^{\mathbf{x}} + \left(\frac{16G^4}{\kappa} - \frac{\kappa}{4}\right) S_{T,i^\star}^{\mathbf{x}} + \frac{1}{2}\kappa D^2 + \mathcal{O}(1), \qquad \text{(D.5)}
\end{aligned}$$

where the last step is due to $\log(1+x) \le x$ for $x \ge 0$.

For *convex* functions, by Lemma 21, the convex base learner guarantees the following:

$$\begin{aligned}
\textsc{Base-Reg} &\le 5D\sqrt{1 + \bar{V}_{T,i^\star}^{\mathrm{c}}} + \kappa D^2 - \frac{1}{4}\kappa S_{T,i^\star}^{\mathbf{x}} + \mathcal{O}(1) \\
&\le 5D\sqrt{1 + 16LF_T^{\mathbf{x}}} + \kappa D^2 - \frac{1}{4}\kappa S_{T,i^\star}^{\mathbf{x}} + \mathcal{O}(1). \qquad \text{(D.6)}
\end{aligned}$$

Overall Regret Analysis. For λ-*strongly convex* functions, combining Eq. (C.5) and Eq. (D.4), we obtain

$$\begin{aligned}
&\textsc{Reg}_T \le \mathcal{O}\left(\frac{1}{\lambda}\log F_T^{\mathbf{x}}\right) + 64G^2 S_T^{\mathbf{x}} + \left(64G^2 + \gamma^{\text{MID}} - \frac{1}{8}\kappa\right) S_{T,i^\star}^{\mathbf{x}} + \frac{1}{4}\kappa D^2 \\
&\quad - \frac{C_0}{2} S_T^{\text{TOP}} - \gamma^{\text{MID}} \sum_{t=2}^{T}\sum_{j=1}^{M} q_{t,j}^{\text{TOP}} \sum_{i=1}^{N} q_{t,j,i}^{\text{MID}} \|\mathbf{x}_{t,i} - \mathbf{x}_{t-1,i}\|^2 - \gamma^{\text{TOP}} \sum_{t=2}^{T}\sum_{j=1}^{M} q_{t,j}^{\text{TOP}} \|\boldsymbol{q}_{t,j}^{\text{MID}} - \boldsymbol{q}_{t-1,j}^{\text{MID}}\|_1^2 \\
&\le \mathcal{O}\left(\frac{1}{\lambda}\log F_T^{\mathbf{x}}\right) + \left(64G^2 + \gamma^{\text{MID}} - \frac{1}{8}\kappa\right) S_{T,i^\star}^{\mathbf{x}} + \frac{1}{4}\kappa D^2 \\
&\qquad + \left(256D^2G^2 - \frac{C_0}{2}\right) S_T^{\text{TOP}} + \left(128G^2 - \gamma^{\text{MID}}\right) \sum_{t=2}^{T}\sum_{j=1}^{M} q_{t,j}^{\text{TOP}} \sum_{i=1}^{N} q_{t,j,i}^{\text{MID}} \|\mathbf{x}_{t,i} - \mathbf{x}_{t-1,i}\|^2 \\
&\qquad + \left(256D^2G^2 - \gamma^{\text{TOP}}\right) \sum_{t=2}^{T}\sum_{j=1}^{M} q_{t,j}^{\text{TOP}} \|\boldsymbol{q}_{t,j}^{\text{MID}} - \boldsymbol{q}_{t-1,j}^{\text{MID}}\|_1^2 \le \mathcal{O}\left(\frac{1}{\lambda}\log F_T^{\mathbf{x}}\right) \le \mathcal{O}\left(\frac{1}{\lambda}\log F_T\right),
\end{aligned}$$

where the second step follows from Lemma 5, the third step requires $\gamma^{\text{TOP}} \ge 256D^2G^2$, $\gamma^{\text{MID}} \ge 128G^2$, $\kappa \ge 512G^2 + 8\gamma^{\text{MID}}$, and $C_0 \ge 512D^2G^2$, and the last step uses Lemma 16 by choosing a, b, c as some T-independent constants and setting

$$x = \sum_{t=1}^{T} f_t(\mathbf{x}_t) - \sum_{t=1}^{T} \min_{\mathbf{x}\in\mathcal{X}_+} f_t(\mathbf{x}), \text{ and } d = \min_{\mathbf{x}\in\mathcal{X}_+} \sum_{t=1}^{T} f_t(\mathbf{x}) - \sum_{t=1}^{T} \min_{\mathbf{x}\in\mathcal{X}_+} f_t(\mathbf{x}).$$

For α-*exp-concave* functions, combining Eq. (C.6) and Eq. (D.5), we obtain

$$\begin{aligned}
&\textsc{Reg}_T \le \mathcal{O}\left(\frac{d}{\alpha}\log F_T^{\mathbf{x}}\right) + \frac{16G^4}{\kappa} S_T^{\mathbf{x}} + \left(\frac{16G^4}{\kappa} + \gamma^{\text{MID}} - \frac{\kappa}{4}\right) S_{T,i^\star}^{\mathbf{x}} + \frac{1}{2}\kappa D^2 \\
&\quad - \frac{C_0}{2} S_T^{\text{TOP}} - \gamma^{\text{MID}} \sum_{t=2}^{T}\sum_{j=1}^{M} q_{t,j}^{\text{TOP}} \sum_{i=1}^{N} q_{t,j,i}^{\text{MID}} \|\mathbf{x}_{t,i} - \mathbf{x}_{t-1,i}\|^2 - \gamma^{\text{TOP}} \sum_{t=2}^{T}\sum_{j=1}^{M} q_{t,j}^{\text{TOP}} \|\boldsymbol{q}_{t,j}^{\text{MID}} - \boldsymbol{q}_{t-1,j}^{\text{MID}}\|_1^2
\end{aligned}$$

$$
\begin{aligned}
&\leq \mathcal{O}\left(\frac{d}{\alpha}\log F_T^{\mathbf{x}}\right) + \left(\frac{16G^4}{\kappa} + \gamma^{\text{MID}} - \frac{\kappa}{4}\right) S_{T,i^\star}^{\mathbf{x}} + \frac{1}{2}\kappa D^2 \\
&\quad + \left(\frac{64D^2G^4}{\kappa} - \frac{C_0}{2}\right) S_T^{\text{TOP}} + \left(\frac{32G^4}{\kappa} - \gamma^{\text{MID}}\right) \sum_{t=2}^{T}\sum_{j=1}^{M} q_{t,j}^{\text{TOP}} \sum_{i=1}^{N} q_{t,j,i}^{\text{MID}} \left\|\mathbf{x}_{t,i} - \mathbf{x}_{t-1,i}\right\|^2 \\
&\quad + \left(\frac{64D^2G^4}{\kappa} - \gamma^{\text{TOP}}\right) \sum_{t=2}^{T}\sum_{j=1}^{M} q_{t,j}^{\text{TOP}} \|\boldsymbol{q}_{t,j}^{\text{MID}} - \boldsymbol{q}_{t-1,j}^{\text{MID}}\|_1^2 \leq \mathcal{O}\left(\frac{d}{\alpha}\log F_T^{\mathbf{x}}\right) \leq \mathcal{O}\left(\frac{d}{\alpha}\log F_T\right),
\end{aligned}
$$

where the second step follows from Lemma 5 and the third step requires $\gamma^{\text{TOP}} \geq 64D^2G^4$, $\gamma^{\text{MID}} \geq 32G^4$, $\kappa \geq 64G^4 + 4\gamma^{\text{MID}}$, and $C_0 \geq 128D^2G^4$. Similar to the *strongly convex* case, the final step follows from Lemma 16, where we choose a, b, c as some T-independent constants, and set x and d to the same values as in the *strongly convex* case.

For *convex* functions, combining Eq. (D.2) and Eq. (D.6), we obtain

$$
\begin{aligned}
\text{Reg}_T &\leq \frac{Z}{\varepsilon_{j^\star}^{\text{TOP}}} \log \frac{N}{3C_0^2(\varepsilon_{j^\star}^{\text{TOP}})^2} + \frac{1024D^2L\varepsilon_{j^\star}^{\text{TOP}}}{Z} F_T^{\mathbf{x}} + 5D\sqrt{1 + 16LF_T^{\mathbf{x}}} \\
&\quad + \left(\frac{64G^2}{Z} + \gamma^{\text{MID}} - \frac{\kappa}{4}\right) S_{T,i^\star}^{\mathbf{x}} + \kappa D^2 \\
&\leq \frac{Z}{\varepsilon_{j^\star}^{\text{TOP}}} \log \frac{N}{3C_0^2(\varepsilon_{j^\star}^{\text{TOP}})^2} + \frac{1024D^2L\varepsilon_{j^\star}^{\text{TOP}}}{Z} F_T^{\mathbf{x}} + 5D\sqrt{1 + 16LF_T^{\mathbf{x}}} + \kappa D^2 \\
&\leq \frac{Z}{\varepsilon_{j^\star}^{\text{TOP}}} \log \frac{Ne^5}{3C_0^2(\varepsilon_{j^\star}^{\text{TOP}})^2} + \left(\frac{1024}{Z} + 20\right) D^2L\varepsilon_{j^\star}^{\text{TOP}} F_T^{\mathbf{x}} + \kappa D^2 + \mathcal{O}(1) \\
&\leq \frac{2Z}{\varepsilon_{j^\star}^{\text{TOP}}} \log \frac{Ne^5}{3C_0^2(\varepsilon_{j^\star}^{\text{TOP}})^2} + \left(\frac{2048}{Z} + 40\right) D^2L\varepsilon_{j^\star}^{\text{TOP}} F_T + \kappa D^2 + \mathcal{O}(1) \leq \mathcal{O}\left(\sqrt{F_T \log F_T}\right),
\end{aligned}
$$

where the second step requires $\kappa \geq \frac{256G^2}{Z} + 4\gamma^{\text{MID}}$, the third step follows from the AM-GM inequality: $\sqrt{ab} \leq \frac{a}{2} + \frac{b}{2}$ for any $a, b > 0$ with $a = 1/(2D\varepsilon_{j^\star}^{\text{TOP}})$ and $b = 2D\varepsilon_{j^\star}^{\text{TOP}} LF_T^{\mathbf{x}}$, the fourth step follows from

$$
x - d \leq c(x - b) + e \Rightarrow x - d \leq \frac{c(d - b) + e}{1 - c} \tag{D.7}
$$

for $x, b, d, e \geq 0$ and $0 < c < 1$ where we choose $x = \sum_{t=1}^T f_t(\mathbf{x}_t)$, $d = \min_{\mathbf{x}\in\mathcal{X}} \sum_{t=1}^T f_t(\mathbf{x})$, $b = \sum_{t=1}^T \min_{\mathbf{x}\in\mathcal{X}_+} f_t(\mathbf{x})$, $c = \left(\frac{1024}{Z} + 20\right) D^2L\varepsilon_{j^\star}^{\text{TOP}} \leq 1/2$, and $e = \mathcal{O}(1)$, and the final step is due to Lemma 12. ■

D.1.2 Gradient-variance Regret

Proof We adopt the same regret decomposition strategy as utilized in Appendix C.3.

Meta Regret Analysis. Recall that the normalization factor $Z = \max\{GD + \gamma^{\text{MID}}D^2, 1 + \gamma^{\text{MID}}D^2 + 2\gamma^{\text{TOP}}\}$. For *strongly convex* and *exp-concave* functions, we follow the same meta regret analysis as used in Appendix C.3.

For *convex* functions, we give a different analysis for $V_\star = \sum_{t=2}^T (\ell^{\text{MID}}_{t,j^\star,i^\star} - m^{\text{MID}}_{t,j^\star,i^\star})^2$. From Lemma 3, it holds that

$$\begin{aligned} V_\star &\le \frac{2D^2}{Z^2} \sum_{t=1}^T \|\mathbf{g}_t - \mathbf{g}_{t-1}\|^2 + \frac{2G^2}{Z^2} \sum_{t=1}^T \|\mathbf{x}_{t,i^\star} - \mathbf{x}_{t-1,i^\star} + \mathbf{x}_{t-1} - \mathbf{x}_t\|^2 \\ &\le \frac{8D^2}{Z^2} W_T + \frac{4G^2}{Z^2} S^{\mathbf{x}}_{T,i^\star} + \frac{4G^2}{Z^2} S^{\mathbf{x}}_T. \end{aligned}$$

Similar to Eq. (D.2), the meta regret can be bounded as

$$\begin{aligned} \textsc{Meta-Reg} &\le \frac{Z}{\varepsilon^{\text{TOP}}_{j^\star}} \log \frac{N}{3C_0^2 (\varepsilon^{\text{TOP}}_{j^\star})^2} + 32 Z \varepsilon^{\text{TOP}}_{j^\star} V_\star + \gamma^{\text{MID}} S^{\mathbf{x}}_{T,i^\star} - \frac{C_0}{2} S^{\text{TOP}}_T \\ &\quad - \gamma^{\text{MID}} \sum_{t=2}^T \sum_{j=1}^M q^{\text{TOP}}_{t,j} \sum_{i=1}^N q^{\text{MID}}_{t,j,i} \|\mathbf{x}_{t,i} - \mathbf{x}_{t-1,i}\|^2 - \gamma^{\text{TOP}} \sum_{t=2}^T \sum_{j=1}^M q^{\text{TOP}}_{t,j} \|\boldsymbol{q}^{\text{MID}}_{t,j} - \boldsymbol{q}^{\text{MID}}_{t-1,j}\|_1^2 \\ &\le 2ZC_0 \log \frac{4N}{3} + 64D \sqrt{W_T \log\left(\frac{1024 N D^2 W_T}{Z^2}\right)} + \left(\frac{64G^2}{Z} + \gamma^{\text{MID}}\right) S^{\mathbf{x}}_{T,i^\star} \\ &\quad + \left(\frac{256 D^2 G^2}{Z} - \frac{C_0}{2}\right) S^{\text{TOP}}_T + \left(\frac{128 G^2}{Z} - \gamma^{\text{MID}}\right) \sum_{t=2}^T \sum_{j=1}^M q^{\text{TOP}}_{t,j} \sum_{i=1}^N q^{\text{MID}}_{t,j,i} \|\mathbf{x}_{t,i} - \mathbf{x}_{t-1,i}\|^2 \\ &\quad + \left(\frac{256 D^2 G^2}{Z} - \gamma^{\text{TOP}}\right) \sum_{t=2}^T \sum_{j=1}^M q^{\text{TOP}}_{t,j} \|\boldsymbol{q}^{\text{MID}}_{t,j} - \boldsymbol{q}^{\text{MID}}_{t-1,j}\|_1^2 \\ &\le 2ZC_0 \log \frac{4N}{3} + \mathcal{O}\left(\sqrt{W_T \log W_T}\right) + \left(\frac{64G^2}{Z} + \gamma^{\text{MID}}\right) S^{\mathbf{x}}_{T,i^\star}, \end{aligned} \tag{D.8}$$

where the final step requires $\gamma^{\text{TOP}} \ge \frac{256D^2G^2}{Z}, \gamma^{\text{MID}} \ge \frac{128G^2}{Z}$, and $C_0 \ge \frac{512D^2G^2}{Z}$.

Base Regret Analysis. We first provide different decompositions of the empirical gradient variation defined on surrogates for strongly convex, exp-concave, and convex functions, respectively, and then analyze the base regret in the corresponding cases.

For λ-*strongly convex* functions, $\bar{V}^{\text{sc}}_{T,i} \triangleq \sum_{t=2}^T \|\nabla h^{\text{sc}}_{t,i^\star}(\mathbf{x}_{t,i^\star}) - \nabla h^{\text{sc}}_{t-1,i^\star}(\mathbf{x}_{t-1,i^\star})\|^2$ satisfies

$$\begin{aligned} \bar{V}^{\text{sc}}_{T,i^\star} &= \sum_{t=2}^T \|(\mathbf{g}_t + \lambda_{i^\star}(\mathbf{x}_{t,i^\star} - \mathbf{x}_t)) - (\mathbf{g}_{t-1} + \lambda_{i^\star}(\mathbf{x}_{t-1,i^\star} - \mathbf{x}_{t-1}))\|^2 \\ &\le 8W_T + 4S^{\mathbf{x}}_T + 4S^{\mathbf{x}}_{T,i^\star}. \end{aligned}$$

For α-*exp-concave* functions, $\bar{V}^{\exp}_{T,i^\star} \triangleq \sum_{t=2}^T \|\nabla h^{\exp}_{t,i^\star}(\mathbf{x}_{t,i^\star}) - \nabla h^{\exp}_{t-1,i^\star}(\mathbf{x}_{t-1,i^\star})\|^2$ satisfies

$$\begin{aligned} \bar{V}^{\exp}_{T,i^\star} &\le 2\sum_{t=2}^T \|\mathbf{g}_t - \mathbf{g}_{t-1}\|^2 + 2\alpha_{i^\star}^2 \sum_{t=2}^T \|\mathbf{g}_t \mathbf{g}_t^\top (\mathbf{x}_{t,i^\star} - \mathbf{x}_t) - \mathbf{g}_{t-1}\mathbf{g}_{t-1}^\top (\mathbf{x}_{t-1,i^\star} - \mathbf{x}_{t-1})\|^2 \\ &\le 8W_T + 4D^2 \sum_{t=2}^T \|\mathbf{g}_t \mathbf{g}_t^\top - \mathbf{g}_{t-1}\mathbf{g}_{t-1}^\top\|^2 + 4G^4 \sum_{t=2}^T \|(\mathbf{x}_{t,i^\star} - \mathbf{x}_t) - (\mathbf{x}_{t-1,i^\star} - \mathbf{x}_{t-1})\|^2 \\ &\qquad\qquad (\text{by } \alpha \in [1/T, 1]) \\ &\le C_{27} W_T + 8G^4 S^{\mathbf{x}}_T + 8G^4 S^{\mathbf{x}}_{T,i^\star}, \end{aligned}$$

where the first step uses the definition of $\nabla h_{t,i}^{\exp}(\mathbf{x}) = \mathbf{g}_t + \alpha_i \mathbf{g}_t \mathbf{g}_t^\top (\mathbf{x} - \mathbf{x}_t)$ and the last step holds by setting $C_{27} = 8 + 64D^2G^2$. For *convex* functions, $\bar{V}_{T,i^\star}^{\mathrm{c}}$ can be bounded by $\bar{V}_{T,i^\star}^{\mathrm{c}} \triangleq \sum_{t=2}^T \|\mathbf{g}_t - \mathbf{g}_{t-1}\|^2 \le 4W_T$. To conclude, for different curvature types, we provide correspondingly different analysis of the empirical gradient variation on surrogates:

$$\bar{V}_{T,i^\star}^{\{\mathrm{sc},\exp,\mathrm{c}\}} \le \begin{cases} 8W_T + 4S_T^{\mathbf{x}} + 4S_{T,i^\star}^{\mathbf{x}}, & \text{when } \{f_t\}_{t=1}^T \text{ are } \lambda\text{-strongly convex}, \\ C_{27}W_T + 8G^4 S_T^{\mathbf{x}} + 8G^4 S_{T,i^\star}^{\mathbf{x}}, & \text{when } \{f_t\}_{t=1}^T \text{ are } \alpha\text{-exp-concave}, \\ 4W_T, & \text{when } \{f_t\}_{t=1}^T \text{ are convex.} \end{cases} \tag{D.9}$$

In the following, we analyze the base regret for different curvature types. For λ-*strongly convex* functions, by Lemma 19, the $i^\star$-th base learner guarantees the following:

$$\begin{aligned} \textsc{Base-Reg} &\le \frac{16G^2}{\lambda_{i^\star}} \log(1 + 8\lambda_{i^\star} W_T + 4\lambda_{i^\star} S_T^{\mathbf{x}} + 4\lambda_{i^\star} S_{T,i^\star}^{\mathbf{x}}) + \frac{1}{4}\kappa D^2 - \frac{1}{8}\kappa S_{T,i^\star}^{\mathbf{x}} + \mathcal{O}(1) \\ &\le \frac{32G^2}{\lambda} \log(1 + 8\lambda W_T) + 64G^2 S_T^{\mathbf{x}} + 64G^2 S_{T,i^\star}^{\mathbf{x}} + \frac{1}{4}\kappa D^2 - \frac{1}{8}\kappa S_{T,i^\star}^{\mathbf{x}} + \mathcal{O}(1), \end{aligned} \tag{D.10}$$

where the last step is due to $\log(1+x) \le x$ for $x \ge 0$. For α-*exp-concave* functions, by Lemma 20, the $i^\star$-th base learner guarantees the following:

$$\begin{aligned} \textsc{Base-Reg} &\le \frac{16d}{\alpha_{i^\star}} \log\left(1 + \frac{\alpha_{i^\star} C_{27}}{8\kappa d} W_T + \frac{\alpha_{i^\star} G^4}{\kappa d} S_T^{\mathbf{x}} + \frac{\alpha_{i^\star} G^4}{\kappa d} S_{T,i^\star}^{\mathbf{x}}\right) + \frac{1}{2}\kappa D^2 - \frac{1}{4}\kappa S_{T,i^\star}^{\mathbf{x}} + \mathcal{O}(1) \\ &\le \frac{32d}{\alpha} \log\left(1 + \frac{\alpha C_{27}}{8\kappa d} W_T\right) + \frac{16G^4}{\kappa} S_T^{\mathbf{x}} + \left(\frac{16G^4}{\kappa} - \frac{\kappa}{4}\right) S_{T,i^\star}^{\mathbf{x}} + \frac{1}{2}\kappa D^2 + \mathcal{O}(1), \end{aligned} \tag{D.11}$$

where the last step is due to $\log(1+x) \le x$ for $x \ge 0$. For *convex* functions, by Lemma 21, the convex base learner guarantees the following:

$$\begin{aligned} \textsc{Base-Reg} &\le 5D\sqrt{1 + \bar{V}_{T,i^\star}^{\mathrm{c}}} + \kappa D^2 - \frac{1}{4}\kappa S_{T,i^\star}^{\mathbf{x}} + \mathcal{O}(1) \\ &\le 5D\sqrt{1 + 4W_T} + \kappa D^2 - \frac{1}{4}\kappa S_{T,i^\star}^{\mathbf{x}} + \mathcal{O}(1). \end{aligned} \tag{D.12}$$

Overall Regret Analysis. For λ-*strongly convex* functions, combining (C.5) and (D.10),

$$\begin{aligned} \textsc{Reg}_T &\le \mathcal{O}\left(\frac{1}{\lambda}\log W_T\right) + 64G^2 S_T^{\mathbf{x}} + \left(64G^2 + \gamma^{\text{MID}} - \frac{1}{8}\kappa\right) S_{T,i^\star}^{\mathbf{x}} + \frac{1}{4}\kappa D^2 \\ &\quad - \frac{C_0}{2} S_T^{\text{TOP}} - \gamma^{\text{MID}} \sum_{t=2}^T \sum_{j=1}^M q_{t,j}^{\text{TOP}} \sum_{i=1}^N q_{t,j,i}^{\text{MID}} \|\mathbf{x}_{t,i} - \mathbf{x}_{t-1,i}\|^2 - \gamma^{\text{TOP}} \sum_{t=2}^T \sum_{j=1}^M q_{t,j}^{\text{TOP}} \|\boldsymbol{q}_{t,j}^{\text{MID}} - \boldsymbol{q}_{t-1,j}^{\text{MID}}\|_1^2 \\ &\le \mathcal{O}\left(\frac{1}{\lambda}\log W_T\right) + \left(64G^2 + \gamma^{\text{MID}} - \frac{1}{8}\kappa\right) S_{T,i^\star}^{\mathbf{x}} + \frac{1}{4}\kappa D^2 \\ &\quad + \left(256D^2G^2 - \frac{C_0}{2}\right) S_T^{\text{TOP}} + \left(128G^2 - \gamma^{\text{MID}}\right) \sum_{t=2}^T \sum_{j=1}^M q_{t,j}^{\text{TOP}} \sum_{i=1}^N q_{t,j,i}^{\text{MID}} \|\mathbf{x}_{t,i} - \mathbf{x}_{t-1,i}\|^2 \\ &\quad + \left(256D^2G^2 - \gamma^{\text{TOP}}\right) \sum_{t=2}^T \sum_{j=1}^M q_{t,j}^{\text{TOP}} \|\boldsymbol{q}_{t,j}^{\text{MID}} - \boldsymbol{q}_{t-1,j}^{\text{MID}}\|_1^2 \le \mathcal{O}\left(\frac{1}{\lambda}\log W_T\right), \end{aligned}$$

where the second step follows from Lemma 5 and the last step requires $\gamma^{\text{TOP}} \geq 256D^2G^2$, $\gamma^{\text{MID}} \geq 128G^2$, $\kappa \geq 512G^2 + 8\gamma^{\text{MID}}$, and $C_0 \geq 512D^2G^2$.

For *α-exp-concave* functions, combining (C.6) and (D.11), we obtain

$$\begin{aligned}
\text{Reg}_T \leq{} & \mathcal{O}\left(\frac{d}{\alpha}\log W_T\right) + \frac{16G^4}{\kappa} S_T^{\mathbf{x}} + \left(\frac{16G^4}{\kappa} + \gamma^{\text{MID}} - \frac{\kappa}{4}\right) S_{T,i^\star}^{\mathbf{x}} + \frac{1}{2}\kappa D^2 \\
& - \frac{C_0}{2} S_T^{\text{TOP}} - \gamma^{\text{MID}} \sum_{t=2}^{T}\sum_{j=1}^{M} q_{t,j}^{\text{TOP}} \sum_{i=1}^{N} q_{t,j,i}^{\text{MID}} \|\mathbf{x}_{t,i} - \mathbf{x}_{t-1,i}\|^2 - \gamma^{\text{TOP}} \sum_{t=2}^{T}\sum_{j=1}^{M} q_{t,j}^{\text{TOP}} \|\boldsymbol{q}_{t,j}^{\text{MID}} - \boldsymbol{q}_{t-1,j}^{\text{MID}}\|_1^2 \\
\leq{} & \mathcal{O}\left(\frac{d}{\alpha}\log W_T\right) + \left(\frac{16G^4}{\kappa} + \gamma^{\text{MID}} - \frac{\kappa}{4}\right) S_{T,i^\star}^{\mathbf{x}} + \frac{1}{2}\kappa D^2 \\
& + \left(\frac{64D^2G^4}{\kappa} - \frac{C_0}{2}\right) S_T^{\text{TOP}} + \left(\frac{32G^4}{\kappa} - \gamma^{\text{MID}}\right) \sum_{t=2}^{T}\sum_{j=1}^{M} q_{t,j}^{\text{TOP}} \sum_{i=1}^{N} q_{t,j,i}^{\text{MID}} \|\mathbf{x}_{t,i} - \mathbf{x}_{t-1,i}\|^2 \\
& + \left(\frac{64D^2G^4}{\kappa} - \gamma^{\text{TOP}}\right) \sum_{t=2}^{T}\sum_{j=1}^{M} q_{t,j}^{\text{TOP}} \|\boldsymbol{q}_{t,j}^{\text{MID}} - \boldsymbol{q}_{t-1,j}^{\text{MID}}\|_1^2 \leq \mathcal{O}\left(\frac{d}{\alpha}\log W_T\right),
\end{aligned}$$

where the second step follows from Lemma 5 and the last step requires $\gamma^{\text{TOP}} \geq 64D^2G^4$, $\gamma^{\text{MID}} \geq 32G^4$, $\kappa \geq 64G^4 + 4\gamma^{\text{MID}}$, and $C_0 \geq 128D^2G^4$.

For *convex* functions, combining (D.8) and (D.12), we obtain

$$\text{Reg}_T \leq \mathcal{O}\left(\sqrt{W_T \log W_T}\right) + \left(\frac{64G^2}{Z} + \gamma^{\text{MID}} - \frac{\kappa}{4}\right) S_{T,i^\star}^{\mathbf{x}} + \kappa D^2 \leq \mathcal{O}\left(\sqrt{W_T \log W_T}\right),$$

where the second step requires $\kappa \geq \frac{256G^2}{Z} + 4\gamma^{\text{MID}}$.

At last, we determine the specific values of C_0, γ^{TOP}, and γ^{MID}. These parameters need to satisfy the following requirements:

$$C_0 \geq 1,\ C_0 \geq 8D,\ C_0 \geq 4\gamma^{\text{TOP}},\ C_0 \geq 512D^2G^2,\ C_0 \geq \frac{512D^2G^2}{Z},\ C_0 \geq 128D^2G^4,$$
$$\gamma^{\text{TOP}} \geq 256D^2G^2,\ \gamma^{\text{TOP}} \geq \frac{256D^2G^2}{Z},\ \gamma^{\text{TOP}} \geq 64D^2G^4,\ \gamma^{\text{MID}} \geq 128G^2,$$
$$\gamma^{\text{MID}} \geq \frac{128G^2}{Z},\ \text{and}\ \gamma^{\text{MID}} \geq 32G^4.$$

As a result, we set

$$C_0 = \max\left\{1, 8D, 4\gamma^{\text{TOP}}, 512D^2G^2, 128D^2G^4\right\},$$
$$\gamma^{\text{TOP}} = \max\left\{256D^2G^2, 64D^2G^4\right\}, \quad \gamma^{\text{MID}} = \max\left\{128G^2, 32G^4\right\},$$

where $Z = \max\{GD + \gamma^{\text{MID}}D^2, 1 + \gamma^{\text{MID}}D^2 + 2\gamma^{\text{TOP}}\}$. ■

D.2 Proof of Corollary 2

We prove the small-loss regret guarantees of UniGrad++.Bregman in Appendix D.2.1 and the gradient-variance regret guarantees of UniGrad++.Bregman in Appendix D.2.2.

D.2.1 Small-Loss Regret

Proof We adopt the same regret decomposition strategy as utilized in Appendix C.4.

Meta Regret Analysis. For *strongly convex* and *exp-concave* functions, we follow the same meta regret analysis as used in Appendix C.4.

For *convex* functions, we give a different analysis for $V_\star = \sum_{t=2}^T (\ell^{\text{MID}}_{t,j^\star,i^\star} - m^{\text{MID}}_{t,j^\star,i^\star})^2$. From Lemma 3, it holds that

$$\text{Meta-Reg} \le C_0 \sqrt{4G^2D^2 + \sum_{t=1}^T \langle \nabla f_t(\mathbf{x}_t) - \nabla f_{t-1}(\mathbf{x}_{t-1}), \mathbf{x}_t - \mathbf{x}_{t,i^\star}\rangle^2} + 2GDC_2$$
$$\le C_0\sqrt{4G^2D^2 + D^2\bar{V}_T} + C_2 \le C_0\sqrt{4G^2D^2 + 16D^2LF_T^{\mathbf{x}}} + 2GDC_2, \tag{D.13}$$

where the last step is by the self-bounding property of $\|\nabla f(\mathbf{x})\|_2^2 \le 4L(f(\mathbf{x}) - \min_{\mathbf{x}\in\mathcal{X}_+} f(\mathbf{x}))$ for any $\mathbf{x} \in \mathcal{X}_+$.

Base Regret Analysis. We first provide different decompositions of the empirical gradient variation defined on surrogates for strongly convex, exp-concave, and convex functions, respectively, and then analyze the base regret in the corresponding cases. For λ-*strongly convex* functions, $\bar{V}^{\text{sc}}_{T,i^\star} \triangleq \sum_{t=2}^T \|\nabla h^{\text{sc}}_{t,i^\star}(\mathbf{x}_{t,i^\star}) - \nabla h^{\text{sc}}_{t-1,i^\star}(\mathbf{x}_{t-1,i^\star})\|^2$ can be bounded by

$$\begin{aligned}
\bar{V}^{\text{sc}}_{T,i^\star} &= \sum_{t=2}^T \left\| \mathbf{g}_t + \frac{\lambda_{i^\star}}{2}(\mathbf{x}_{t,i^\star} - \mathbf{x}_t) - \mathbf{g}_{t-1} - \frac{\lambda_{i^\star}}{2}(\mathbf{x}_{t-1,i^\star} - \mathbf{x}_{t-1}) \right\|^2 \\
&\le 3\bar{V}_T + 3\sum_{t=2}^T \left\| \frac{\lambda_{i^\star}}{2}(\mathbf{x}_{t,i^\star} - \mathbf{x}_t)\right\|^2 + 3\sum_{t=2}^T \left\| \frac{\lambda_{i^\star}}{2}(\mathbf{x}_{t-1,i^\star} - \mathbf{x}_{t-1})\right\|^2 \\
&\le 48LF_T^{\mathbf{x}} + 2\lambda_{i^\star}^2 \sum_{t=1}^T \|\mathbf{x}_{t,i^\star} - \mathbf{x}_t\|^2, && \text{(by (6.2))}
\end{aligned}$$

where the first step is due to the property of the surrogate: $\nabla h^{\text{sc}}_{t,i}(\mathbf{x}_{t,i}) = \mathbf{g}_t + \frac{\lambda_i}{2}(\mathbf{x}_{t,i} - \mathbf{x}_t)$, and the second step is due to the Cauchy-Schwarz inequality. For α-*exp-concave* functions, $\bar{V}^{\text{exp}}_{T,i^\star} \triangleq \sum_{t=2}^T \|\nabla h^{\text{exp}}_{t,i^\star}(\mathbf{x}_{t,i^\star}) - \nabla h^{\text{exp}}_{t-1,i^\star}(\mathbf{x}_{t-1,i^\star})\|^2$ can be bounded by

$$\begin{aligned}
\bar{V}^{\text{exp}}_{T,i^\star} &= \sum_{t=2}^T \left\| \mathbf{g}_t + \frac{\alpha_{i^\star}}{2}\mathbf{g}_t\langle \mathbf{g}_t, \mathbf{x}_t - \mathbf{x}_{t,i^\star}\rangle - \mathbf{g}_{t-1} - \frac{\alpha_{i^\star}}{2}\mathbf{g}_{t-1}\langle \mathbf{g}_{t-1}, \mathbf{x}_{t-1} - \mathbf{x}_{t-1,i^\star}\rangle \right\|^2 \\
&\le 3\bar{V}_T + 3\sum_{t=2}^T \left\| \frac{\alpha_{i^\star}}{2}\mathbf{g}_t\langle \mathbf{g}_t, \mathbf{x}_t - \mathbf{x}_{t,i^\star}\rangle \right\|^2 + 3\sum_{t=2}^T \left\| \frac{\alpha_{i^\star}}{2}\mathbf{g}_{t-1}\langle \mathbf{g}_{t-1}, \mathbf{x}_{t-1} - \mathbf{x}_{t-1,i^\star}\rangle \right\|^2 \\
&\le 3\bar{V}_T + 6\sum_{t=1}^T \left\| \frac{\alpha_{i^\star}}{2}\mathbf{g}_t\langle \mathbf{g}_t, \mathbf{x}_t - \mathbf{x}_{t,i^\star}\rangle \right\|^2 \\
&\le 48LF_T^{\mathbf{x}} + 2\alpha_{i^\star}^2 G^2 \sum_{t=1}^T \langle \mathbf{g}_t, \mathbf{x}_t - \mathbf{x}_{t,i^\star}\rangle^2, && \text{(by Assumption 2 and (6.2))}
\end{aligned}$$

where the first step uses the definition of $\nabla h^{\text{exp}}_{t,i}(\mathbf{x}) = \mathbf{g}_t + \frac{\alpha_i}{2}\mathbf{g}_t\mathbf{g}_t^\top(\mathbf{x} - \mathbf{x}_t)$, and the second step is due to the Cauchy-Schwarz inequality. For *convex* functions, the empirical gradient variation $\bar{V}^{\text{c}}_{T,i^\star} \triangleq \sum_{t=2}^T \|\nabla f_t(\mathbf{x}_t) - \nabla f_{t-1}(\mathbf{x}_{t-1})\|^2$ can be bounded by $\bar{V}^{\text{c}}_{T,i^\star} \leq 16LF_T^{\mathbf{x}}$. To conclude, for different curvature types, we provide correspondingly different analysis of the empirical gradient variation on surrogates:

$$\bar{V}^{\{\text{sc,exp,c}\}}_{T,i^\star} \leq \begin{cases} 48LF_T^{\mathbf{x}} + 2\lambda_{i^\star}^2 \sum_{t=1}^T \|\mathbf{x}_{t,i^\star} - \mathbf{x}_t\|^2, & \text{when } \{f_t\}_{t=1}^T \text{ are } \lambda\text{-strongly convex,} \\ 48LF_T^{\mathbf{x}} + 2\alpha_{i^\star}^2 G^2 \sum_{t=1}^T \langle \mathbf{g}_t, \mathbf{x}_t - \mathbf{x}_{t,i^\star}\rangle^2, & \text{when} \{f_t\}_{t=1}^T \text{ are } \alpha\text{-exp-concave,} \\ 16LF_T^{\mathbf{x}}, & \text{when} \{f_t\}_{t=1}^T \text{ are convex.} \end{cases} \tag{D.14}$$

In the following, we analyze the base regret for different curvature types. For λ-*strongly convex* functions, when using the update rule (C.2), according to Lemma 19, the base regret can be bounded as

$$\begin{aligned} &\textsc{Base-Reg} \leq \frac{16G^2}{\lambda_{i^\star}} \log\left(1 + \lambda_{i^\star} \bar{V}^{\text{sc}}_{T,i^\star}\right) + \mathcal{O}(1) \\ \leq{}& \frac{16G^2}{\lambda_{i^\star}} \log\left(1 + 48L\lambda_{i^\star} F_T^{\mathbf{x}} + 2\lambda_{i^\star}^3 \sum_{t=1}^T \|\mathbf{x}_{t,i^\star} - \mathbf{x}_t\|^2\right) + \mathcal{O}(1) \\ \leq{}& \frac{16G^2}{\lambda_{i^\star}} \log\left(C_{18}\left(1 + 48L\lambda_{i^\star} F_T^{\mathbf{x}}\right)\right) + \frac{16G^2}{C_{18}\lambda_{i^\star}} \left(2\lambda_{i^\star}^3 \sum_{t=1}^T \|\mathbf{x}_{t,i^\star} - \mathbf{x}_t\|^2\right) \\ \leq{}& \frac{32G^2}{\lambda} \log(1 + 48LF_T^{\mathbf{x}}) + \frac{32G^2}{C_{18}} \sum_{t=1}^T \|\mathbf{x}_{t,i^\star} - \mathbf{x}_t\|^2 + \mathcal{O}(\log C_{18}), \end{aligned}$$

where the third step requires $C_{18} \geq 1$ by Lemma 15 and uses the property of the best base learner, i.e., $\lambda_{i^\star} \leq \lambda \leq 2\lambda_{i^\star}$. The last step is due to $\lambda_i \leq 1$. For α-*exp-concave* functions, by Lemma 20, the $i^\star$-th base learner guarantees the following:

$$\begin{aligned} &\textsc{Base-Reg} \leq \frac{16d}{\alpha_{i^\star}} \log\left(1 + \frac{\alpha_{i^\star}}{8d} \bar{V}^{\text{exp}}_{T,i^\star}\right) + \mathcal{O}(1) \\ \leq{}& \frac{16d}{\alpha_{i^\star}} \log\left(1 + \frac{6L\alpha_{i^\star}}{d} F_T^{\mathbf{x}} + \frac{\alpha_{i^\star}^3 G^2}{4d} \sum_{t=1}^T \langle \mathbf{g}_t, \mathbf{x}_t - \mathbf{x}_{t,i^\star}\rangle^2\right) + \mathcal{O}(1) \\ \leq{}& \frac{16d}{\alpha_{i^\star}} \log\left(C_{19}\left(1 + \frac{6L\alpha_{i^\star}}{d} F_T^{\mathbf{x}}\right)\right) + \frac{16d}{C_{19}\alpha_{i^\star}} \left(\frac{\alpha_{i^\star}^3 G^2}{4d} \sum_{t=1}^T \langle \mathbf{g}_t, \mathbf{x}_t - \mathbf{x}_{t,i^\star}\rangle^2\right) \\ \leq{}& \frac{32d}{\alpha} \log\left(1 + \frac{6L}{d} F_T^{\mathbf{x}}\right) + \frac{4G^2}{C_{19}} \sum_{t=1}^T \langle \mathbf{g}_t, \mathbf{x}_t - \mathbf{x}_{t,i^\star}\rangle^2 + \mathcal{O}(\log C_{19}), \end{aligned}$$

where the third step requires $C_{19} \geq 1$ by Lemma 15 and uses the property of the best base learner, i.e., $\alpha_{i^\star} \leq \alpha \leq 2\alpha_{i^\star}$. The last step is due to $\alpha_i \leq 1$. For *convex* functions, by Lemma 21, the base regret can be bounded as

$$\textsc{Base-Reg} \leq 5D\sqrt{1 + \bar{V}_T} + \mathcal{O}(1) \leq 5D\sqrt{1 + 16LF_T^{\mathbf{x}}} + \mathcal{O}(1).$$

Overall Regret Analysis. For λ-*strongly convex* functions, by combining the meta and base regret, it holds that

$$\begin{aligned}\textsc{Reg}_T &\le \left(\frac{C_0 D^2}{2C_3} + \frac{32G^2}{C_{18}} - \frac{\lambda_{i^\star}}{4}\right)\sum_{t=1}^T \|\mathbf{x}_t - \mathbf{x}_{t,i^\star}\|^2 + \frac{32G^2}{\lambda}\log(1+48LF_T^{\mathbf{x}}) + \mathcal{O}(C_3 + \log C_{18}) \\ &\le \frac{32G^2}{\lambda}\log(1+48LF_T^{\mathbf{x}}) \le \mathcal{O}\left(\frac{1}{\lambda}\log F_T\right),\end{aligned}$$

where the second step is by choosing $C_3 = 4C_0D^2/\lambda_{i^\star}$ and $C_{18} = \max\{1, 256G^2/\lambda_{i^\star}\}$ and the last step is due to Lemma 16 by choosing a, b, c as some T-independent constants and setting $x = \sum_{t=1}^T f_t(\mathbf{x}_t) - \sum_{t=1}^T \min_{\mathbf{x}\in\mathcal{X}_+} f_t(\mathbf{x})$ and $d = \min_{\mathbf{x}\in\mathcal{X}} \sum_{t=1}^T f_t(\mathbf{x}) - \sum_{t=1}^T \min_{\mathbf{x}\in\mathcal{X}_+} f_t(\mathbf{x})$. Note that such a parameter configuration will only add an $\mathcal{O}(1/\lambda)$ factor to the final regret bound, which can be absorbed.

For α-*exp-concave* functions, by combining the meta and base regret, it holds that

$$\begin{aligned}\textsc{Reg}_T &\le \left(\frac{C_0}{2C_4} + \frac{4G^2}{C_{19}} - \frac{\alpha_{i^\star}}{4}\right)\sum_{t=1}^T \langle \mathbf{g}_t, \mathbf{x}_t - \mathbf{x}_{t,i^\star}\rangle^2 + \frac{32d}{\alpha}\log\left(1 + \frac{6L}{d}F_T^{\mathbf{x}}\right) + \mathcal{O}(C_4 + \log C_{19}) \\ &\le \frac{32d}{\alpha}\log\left(1+\frac{6L}{d}F_T^{\mathbf{x}}\right) + \mathcal{O}(1) \le \mathcal{O}\left(\frac{d}{\alpha}\log F_T\right),\end{aligned}$$

where the second step chooses $C_4 = 4C_0/\alpha_{i^\star}$ and $C_{19} = \max\{1, 32G^2/\alpha_{i^\star}\}$. Similar to the *strongly convex* case, the final step follows from Lemma 16, where we choose a, b, c as some T-independent constants, and set x and d to the same values as in the *strongly convex* case. Meanwhile, such a parameter configuration will only add an $\mathcal{O}(1/\alpha)$ factor to the final regret bound, which can be absorbed.

For *convex* functions, by combining the meta and base regret, it holds that

$$\textsc{Reg}_T \le C_0\sqrt{4G^2D^2 + 16D^2LF_T^{\mathbf{x}}} + 5D\sqrt{1+16LF_T^{\mathbf{x}}} + 2GDC_2 + \mathcal{O}(1) \le \mathcal{O}(\sqrt{F_T}),$$

where the final step follows from Lemma 17 by setting a, b as some T-independent constants and choosing $x = \sum_{t=1}^T f_t(\mathbf{x}_t) - \sum_{t=1}^T \min_{\mathbf{x}\in\mathcal{X}_+} f_t(\mathbf{x})$ and $d = \min_{\mathbf{x}\in\mathcal{X}} \sum_{t=1}^T f_t(\mathbf{x}) - \sum_{t=1}^T \min_{\mathbf{x}\in\mathcal{X}_+} f_t(\mathbf{x})$. ■

D.2.2 Gradient-variance Regret

Proof We adopt the same regret decomposition strategy as utilized in Appendix C.4.

Meta Regret Analysis. For *strongly convex* and *exp-concave* functions, we follow the same meta regret analysis as used in Appendix C.4.

For *convex* functions, we give a different analysis for $V_\star = \sum_{t=2}^T (\ell^{\text{MID}}_{t,j^\star,i^\star} - m^{\text{MID}}_{t,j^\star,i^\star})^2$. From Lemma 3, it holds that

$$\begin{aligned}\textsc{Meta-Reg} &\le C_0\sqrt{4G^2D^2 + \sum_{t=1}^T \langle \nabla f_t(\mathbf{x}_t) - \nabla f_{t-1}(\mathbf{x}_{t-1}), \mathbf{x}_t - \mathbf{x}_{t,i^\star}\rangle^2} + 2GDC_2 \\ &\le C_0\sqrt{1 + D^2\bar{V}_T} + 2GDC_2 \le C_0\sqrt{1+4D^2W_T} + 2GDC_2. \end{aligned} \tag{D.15}$$

Base Regret Analysis. We first provide different decompositions of the empirical gradient variation defined on surrogates for strongly convex, exp-concave, and convex functions, respectively, and then analyze the base regret in the corresponding cases. For λ-*strongly convex* functions, $\bar{V}_{T,i}^{\text{sc}} \triangleq \sum_{t=2}^{T} \|\nabla h_{t,i}^{\text{sc}}(\mathbf{x}_{t,i^\star}) - \nabla h_{t-1,i}^{\text{sc}}(\mathbf{x}_{t-1,i^\star})\|^2$ can be bounded by

$$\begin{aligned}
\bar{V}_{T,i^\star}^{\text{sc}} &= \sum_{t=2}^{T} \left\| \mathbf{g}_t + \frac{\lambda_{i^\star}}{2}(\mathbf{x}_{t,i^\star} - \mathbf{x}_t) - \mathbf{g}_{t-1} - \frac{\lambda_{i^\star}}{2}(\mathbf{x}_{t-1,i^\star} - \mathbf{x}_{t-1}) \right\|^2 \\
&\le 3\bar{V}_T + 3\sum_{t=2}^{T} \left\| \frac{\lambda_{i^\star}}{2}(\mathbf{x}_{t,i^\star} - \mathbf{x}_t) \right\|^2 + 3\sum_{t=2}^{T} \left\| \frac{\lambda_{i^\star}}{2}(\mathbf{x}_{t-1,i^\star} - \mathbf{x}_{t-1}) \right\|^2 \\
&\le 12W_T + 2\lambda_{i^\star}^2 \sum_{t=1}^{T} \|\mathbf{x}_{t,i^\star} - \mathbf{x}_t\|^2, \qquad \text{(by (6.3))}
\end{aligned}$$

where the first step is due to the property of the surrogate: $\nabla h_{t,i}^{\text{sc}}(\mathbf{x}_{t,i}) = \mathbf{g}_t + \frac{\lambda_i}{2}(\mathbf{x}_{t,i} - \mathbf{x}_t)$, and the second step is due to the Cauchy-Schwarz inequality. For α-*exp-concave* functions, $\bar{V}_{T,i^\star}^{\text{exp}} \triangleq \sum_{t=2}^{T} \|\nabla h_{t,i^\star}^{\text{exp}}(\mathbf{x}_{t,i^\star}) - \nabla h_{t-1,i^\star}^{\text{exp}}(\mathbf{x}_{t-1,i^\star})\|^2$ can be bounded by

$$\begin{aligned}
\bar{V}_{T,i^\star}^{\text{exp}} &= \sum_{t=2}^{T} \left\| \mathbf{g}_t + \frac{\alpha_{i^\star}}{2}\mathbf{g}_t\langle \mathbf{g}_t, \mathbf{x}_t - \mathbf{x}_{t,i^\star}\rangle - \mathbf{g}_{t-1} - \frac{\alpha_{i^\star}}{2}\mathbf{g}_{t-1}\langle \mathbf{g}_{t-1}, \mathbf{x}_{t-1} - \mathbf{x}_{t-1,i^\star}\rangle \right\|^2 \\
&\le 3\bar{V}_T + 3\sum_{t=2}^{T} \left\| \frac{\alpha_{i^\star}}{2}\mathbf{g}_t\langle \mathbf{g}_t, \mathbf{x}_t - \mathbf{x}_{t,i^\star}\rangle \right\|^2 + 3\sum_{t=2}^{T} \left\| \frac{\alpha_{i^\star}}{2}\mathbf{g}_{t-1}\langle \mathbf{g}_{t-1}, \mathbf{x}_{t-1} - \mathbf{x}_{t-1,i^\star}\rangle \right\|^2 \\
&\le 3\bar{V}_T + 6\sum_{t=1}^{T} \left\| \frac{\alpha_{i^\star}}{2}\mathbf{g}_t\langle \mathbf{g}_t, \mathbf{x}_t - \mathbf{x}_{t,i^\star}\rangle \right\|^2 \\
&\le 12W_T + 2\alpha_{i^\star}^2 G^2 \sum_{t=1}^{T} \langle \mathbf{g}_t, \mathbf{x}_t - \mathbf{x}_{t,i^\star}\rangle^2, \qquad \text{(by Assumption 2 and (6.3))}
\end{aligned}$$

where the first step uses the definition of $\nabla h_{t,i}^{\text{exp}}(\mathbf{x}) = \mathbf{g}_t + \frac{\alpha_i}{2}\mathbf{g}_t\mathbf{g}_t^\top(\mathbf{x} - \mathbf{x}_t)$, and the second step is due to the Cauchy-Schwarz inequality. For *convex* functions, the empirical gradient variation $\bar{V}_{T,i^\star}^{\text{c}} \triangleq \sum_{t=2}^{T} \|\nabla f_t(\mathbf{x}_t) - \nabla f_{t-1}(\mathbf{x}_{t-1})\|^2$ can be bounded by $\bar{V}_{T,i^\star}^{\text{c}} \le 4W_T$. To conclude, for different curvature types, we provide correspondingly different analysis of the empirical gradient variation on surrogates:

$$\bar{V}_{T,i^\star}^{\{\text{sc},\text{exp},\text{c}\}} \le \begin{cases} 12W_T + 2\lambda_{i^\star}^2 \sum_{t=1}^{T} \|\mathbf{x}_{t,i^\star} - \mathbf{x}_t\|^2, & \text{when } \{f_t\}_{t=1}^T \text{ are } \lambda\text{-strongly convex,} \\ 12W_T + 2\alpha_{i^\star}^2 G^2 \sum_{t=1}^{T} \langle \mathbf{g}_t, \mathbf{x}_t - \mathbf{x}_{t,i^\star}\rangle^2, & \text{when} \{f_t\}_{t=1}^T \text{ are } \alpha\text{-exp-concave,} \\ 4W_T, & \text{when} \{f_t\}_{t=1}^T \text{ are convex.} \end{cases} \tag{D.16}$$

In the following, we analyze the base regret for different curvature types. For λ-*strongly convex* functions, when using the update rule (C.2), according to Lemma 19, the base regret can be bounded as

$$\begin{aligned}\textsc{Base-Reg} &\leq \frac{16G^2}{\lambda_{i^\star}} \log\left(1 + 12\lambda_{i^\star} W_T + 2\lambda_{i^\star}^3 \sum_{t=1}^T \|\mathbf{x}_{t,i^\star} - \mathbf{x}_t\|^2\right) + \mathcal{O}(1)\\
&\leq \frac{16G^2}{\lambda_{i^\star}} \log\left(C_{20}\left(1 + 12\lambda_{i^\star} W_T\right)\right) + \frac{16G^2}{C_{20}\lambda_{i^\star}}\left(2\lambda_{i^\star}^3 \sum_{t=1}^T \|\mathbf{x}_{t,i^\star} - \mathbf{x}_t\|^2\right)\\
&\leq \frac{32G^2}{\lambda} \log(1 + 12W_T) + \frac{32G^2}{C_{20}} \sum_{t=1}^T \|\mathbf{x}_{t,i^\star} - \mathbf{x}_t\|^2 + \mathcal{O}(\log C_{20}),\end{aligned}$$

where the second step requires $C_{20} \geq 1$ by Lemma 15 and uses the property of the best base learner, i.e., $\lambda_{i^\star} \leq \lambda \leq 2\lambda_{i^\star}$. The last step is due to $\lambda_i \leq 1$. For *α-exp-concave* functions, by Lemma 20, the $i^\star$-th base learner guarantees the following:

$$\begin{aligned}\textsc{Base-Reg} &\leq \frac{16d}{\alpha_{i^\star}} \log\left(1 + \frac{3\alpha_{i^\star}}{2d} W_T + \frac{\alpha_{i^\star}^3 G^2}{4d} \sum_{t=1}^T \langle \mathbf{g}_t, \mathbf{x}_t - \mathbf{x}_{t,i^\star}\rangle^2\right) + \mathcal{O}(1)\\
&\leq \frac{16d}{\alpha_{i^\star}} \log\left(C_{21}\left(1 + \frac{3\alpha_{i^\star}}{2d} W_T\right)\right) + \frac{16d}{C_{21}\alpha_{i^\star}}\left(\frac{\alpha_{i^\star}^3 G^2}{4d} \sum_{t=1}^T \langle \mathbf{g}_t, \mathbf{x}_t - \mathbf{x}_{t,i^\star}\rangle^2\right)\\
&\leq \frac{32d}{\alpha} \log\left(1 + \frac{3}{2d} W_T\right) + \frac{4G^2}{C_{21}} \sum_{t=1}^T \langle \mathbf{g}_t, \mathbf{x}_t - \mathbf{x}_{t,i^\star}\rangle^2 + \mathcal{O}(\log C_{21}),\end{aligned}$$

where the second step requires $C_{21} \geq 1$ by Lemma 15 and uses the property of the best base learner, i.e., $\alpha_{i^\star} \leq \alpha \leq 2\alpha_{i^\star}$. The last step is due to $\alpha_i \leq 1$. For *convex* functions, by Lemma 21, the base regret can be bounded as

$$\textsc{Base-Reg} \leq 5D\sqrt{1 + \bar{V}_T} + \mathcal{O}(1) \leq 5D\sqrt{1 + 4W_T} + \mathcal{O}(1).$$

Overall Regret Analysis. For *λ-strongly convex* functions, by combining the meta and base regret, it holds that

$$\begin{aligned}\textsc{Reg}_T &\leq \left(\frac{C_0 D^2}{2C_3} + \frac{32G^2}{C_{20}} - \frac{\lambda_{i^\star}}{4}\right) \sum_{t=1}^T \|\mathbf{x}_t - \mathbf{x}_{t,i^\star}\|^2 + \frac{32G^2}{\lambda} \log(1 + 12W_T) + \mathcal{O}(C_3 + \log C_{20})\\
&\leq \frac{32G^2}{\lambda} \log(1 + 12W_T) \leq \mathcal{O}\left(\frac{1}{\lambda} \log W_T\right),\end{aligned}$$

where the second step is by choosing $C_3 = 4C_0 D^2/\lambda_{i^\star}$ and $C_{20} = \max\{1, 256G^2/\lambda_{i^\star}\}$. For *$\alpha$-exp-concave* functions, by combining the meta and base regret, it holds that

$$\begin{aligned}\textsc{Reg}_T &\leq \left(\frac{C_0}{2C_4} + \frac{4G^2}{C_{21}} - \frac{\alpha_{i^\star}}{4}\right) \sum_{t=1}^T \langle \mathbf{g}_t, \mathbf{x}_t - \mathbf{x}_{t,i^\star}\rangle^2 + \frac{32d}{\alpha} \log\left(1 + \frac{3W_T}{2d}\right) + \mathcal{O}(C_4 + \log C_{21})\\
&\leq \frac{32d}{\alpha} \log\left(1 + \frac{3}{2d} W_T\right) + \mathcal{O}(1) \leq \mathcal{O}\left(\frac{d}{\alpha} \log W_T\right),\end{aligned}$$

where the second step chooses $C_4 = 4C_0/\alpha_{i^\star}$ and $C_{21} = \max\{1, 32G^2/\alpha_{i^\star}\}$. For *convex* functions, by combining the meta and base regret, it holds that

$$\textsc{Reg}_T \leq C_0\sqrt{4G^2D^2 + 4D^2 W_T} + 5D\sqrt{4G^2D^2 + 4W_T} + 2GDC_2 + \mathcal{O}(1) \leq \mathcal{O}(\sqrt{W_T}).$$

Finally, we note that the constants $C_3, C_4, C_{18}.C_{19}, C_{20}, C_{21}$ only exist in analysis and and hence our choices of them are feasible.

■

D.3 Proof of Theorem 5

Proof Recall that we denote by $\mathbf{g}_t = \nabla f_t(\mathbf{x}_t)$ for simplicity. To begin with, we provide a proof for Eq. (6.5):

$$\begin{aligned}
\mathbb{E}[\bar{V}_T] = {} & \mathbb{E}\left[\sum_{t=2}^{T} \|\nabla f_t(\mathbf{x}_t) - \nabla f_{t-1}(\mathbf{x}_{t-1})\|_2^2\right] \\
\leq & 4\mathbb{E}\left[\sum_{t=2}^{T} \|\nabla f_t(\mathbf{x}_t) - \nabla F_t(\mathbf{x}_t)\|_2^2\right] + 4\mathbb{E}\left[\sum_{t=2}^{T} \|\nabla F_t(\mathbf{x}_t) - \nabla F_t(\mathbf{x}_{t-1})\|_2^2\right] \\
& + 4\mathbb{E}\left[\sum_{t=2}^{T} \|\nabla F_t(\mathbf{x}_{t-1}) - \nabla F_{t-1}(\mathbf{x}_{t-1})\|_2^2\right] + 4\mathbb{E}\left[\sum_{t=2}^{T} \|\nabla F_{t-1}(\mathbf{x}_{t-1}) - \nabla f_{t-1}(\mathbf{x}_{t-1})\|_2^2\right] \\
\leq & 8\sigma_{1:T}^2 + 4\Sigma_{1:T}^2 + 4L^2\mathbb{E}\left[\sum_{t=2}^{T} \|\mathbf{x}_t - \mathbf{x}_{t-1}\|_2^2\right],
\end{aligned} \tag{D.17}$$

where the second step is due to Cauchy-Schwarz inequality and the last step is because of the definitions of $\sigma_{1:T}^2$ and $\Sigma_{1:T}^2$ (given in Section 6.2).

In the following, we present regret decompositions tailored to different curvature regimes, proceed to analyze both the meta and base regret components, and finally combine these results to derive the overall regret bounds.

Regret Decomposition. For λ-*strongly convex* functions, similar to the decomposition in Appendix C.3, we have

$$\begin{aligned}
& \mathbb{E}[\textsc{Reg}_T] \leq \mathbb{E}\left[\sum_{t=1}^{T} \langle \nabla F_t(\mathbf{x}_t), \mathbf{x}_t - \mathbf{x}^\star \rangle\right] - \frac{\lambda}{2}\mathbb{E}\left[\sum_{t=1}^{T} \|\mathbf{x}_t - \mathbf{x}^\star\|^2\right] \\
& = \mathbb{E}\left[\sum_{t=1}^{T} \langle \mathbf{g}_t, \mathbf{x}_t - \mathbf{x}^\star \rangle\right] - \frac{\lambda}{2}\mathbb{E}\left[\sum_{t=1}^{T} \|\mathbf{x}_t - \mathbf{x}^\star\|^2\right] \\
& \leq \underbrace{\mathbb{E}\left[\sum_{t=1}^{T} \langle \mathbf{g}_t, \mathbf{x}_t - \mathbf{x}_{t,i^\star} \rangle\right] - \frac{\lambda_{i^\star}}{2}\mathbb{E}\left[\sum_{t=1}^{T} \|\mathbf{x}_t - \mathbf{x}_{t,i^\star}\|^2\right]}_{\textsc{Meta-Reg}} + \underbrace{\mathbb{E}\left[\sum_{t=1}^{T} h_{t,i^\star}^{\mathrm{sc}}(\mathbf{x}_{t,i^\star}) - h_{t,i^\star}^{\mathrm{sc}}(\mathbf{x}^\star)\right]}_{\textsc{Base-Reg}},
\end{aligned} \tag{D.18}$$

where the first and second steps rely on the expected loss function $F_t(\mathbf{x}) = \mathbb{E}[f_t(\mathbf{x})]$; in particular, the second step additionally requires that $F_t(\cdot)$ be *strongly convex*.

For α-*exp-concave* functions, following the similar decomposition as in the proof of Theorem 3 in Appendix C.3, we decompose the regret as

$$\mathbb{E}[\textsc{Reg}_T] = \mathbb{E}\left[\sum_{t=1}^{T}\langle \mathbf{g}_t, \mathbf{x}_t - \mathbf{x}^\star\rangle\right] - \frac{\alpha}{2}\mathbb{E}\left[\sum_{t=1}^{T}\langle \mathbf{g}_t, \mathbf{x}_t - \mathbf{x}^\star\rangle^2\right]$$
$$\leq \underbrace{\mathbb{E}\left[\sum_{t=1}^{T}\langle \mathbf{g}_t, \mathbf{x}_t - \mathbf{x}_{t,i^\star}\rangle\right] - \frac{\alpha_{i^\star}}{2}\mathbb{E}\left[\sum_{t=1}^{T}\langle \mathbf{g}_t, \mathbf{x}_t - \mathbf{x}^\star\rangle^2\right]}_{\textsc{Meta-Reg}} + \underbrace{\mathbb{E}\left[\sum_{t=1}^{T} h^{\exp}_{t,i^\star}(\mathbf{x}_{t,i^\star}) - h^{\exp}_{t,i^\star}(\mathbf{x}^\star)\right]}_{\textsc{Base-Reg}}, \tag{D.19}$$

where the first step is due to the exp-concavity and defining surrogate loss functions $h^{\exp}_{t,i}(\mathbf{x}) = \langle \mathbf{g}_t, \mathbf{x}\rangle + \frac{\alpha_i}{2}\langle \mathbf{g}_t, \mathbf{x} - \mathbf{x}_t\rangle^2$. For *convex* functions, we decompose the regret as

$$\mathbb{E}[\textsc{Reg}_T] \leq \underbrace{\mathbb{E}\left[\sum_{t=1}^{T}\langle \mathbf{g}_t, \mathbf{x}_t - \mathbf{x}_{t,i^\star}\rangle\right]}_{\textsc{Meta-Reg}} + \underbrace{\mathbb{E}\left[\sum_{t=1}^{T} h^{\mathrm{c}}_{t,i^\star}(\mathbf{x}_{t,i^\star}) - h^{\mathrm{c}}_{t,i^\star}(\mathbf{x}^\star)\right]}_{\textsc{Base-Reg}}, \tag{D.20}$$

where we have $h^{\mathrm{c}}_{t,i}(\mathbf{x}) = \langle \mathbf{g}_t, \mathbf{x}\rangle$.

Meta Regret Analysis. Recall that the normalization factor $Z = \max\{GD + \gamma^{\text{MID}}D^2, 1 + \gamma^{\text{MID}}D^2 + 2\gamma^{\text{TOP}}\}$. Our Algorithm 4 can be applied to the SEA model without any algorithm modifications. As a result, we directly use the same parameter configurations as in the proof of Theorem 3 (i.e., in Appendix C.3).

For *strongly convex* and *exp-concave* functions, the meta regret is bounded in a similar way as (C.5) and (C.6), and thus omitted here.

For *convex* functions, by Lemma 3, the meta regret can be bounded as

$$\begin{aligned}
\textsc{Meta-Reg} \leq{}& \frac{Z}{\varepsilon^{\text{TOP}}_{j^\star}}\log\frac{N}{3C_0^2(\varepsilon^{\text{TOP}}_{j^\star})^2} + \frac{128D^2\varepsilon^{\text{TOP}}_{j^\star}}{Z}\mathbb{E}[\bar{V}_T] + \left(\frac{64G^2}{Z} + \gamma^{\text{MID}}\right)\mathbb{E}[S^{\mathbf{x}}_{T,i^\star}]\\
&+ \frac{64G^2}{Z}\mathbb{E}[S^{\mathbf{x}}_T] - \frac{C_0}{2}\mathbb{E}[S^{\text{TOP}}_T] - \gamma^{\text{MID}}\mathbb{E}\left[\sum_{t=2}^{T}\sum_{j=1}^{M} q^{\text{TOP}}_{t,j}\sum_{i=1}^{N} q^{\text{MID}}_{t,j,i}\|\mathbf{x}_{t,i} - \mathbf{x}_{t-1,i}\|^2\right]\\
&- \gamma^{\text{TOP}}\mathbb{E}\left[\sum_{t=2}^{T}\sum_{j=1}^{M} q^{\text{TOP}}_{t,j}\|\boldsymbol{q}^{\text{MID}}_{t,j} - \boldsymbol{q}^{\text{MID}}_{t-1,j}\|_1^2\right]\\
\leq{}& \frac{Z}{\varepsilon^{\text{TOP}}_{j^\star}}\log\frac{N}{3C_0^2(\varepsilon^{\text{TOP}}_{j^\star})^2} + \frac{512D^2\varepsilon^{\text{TOP}}_{j^\star}}{Z}\left(2\sigma^2_{1:T} + \Sigma^2_{1:T}\right) + \left(\frac{64G^2}{Z} + \gamma^{\text{MID}}\right)\mathbb{E}[S^{\mathbf{x}}_{T,i^\star}]\\
&- \frac{C_0}{2}\mathbb{E}[S^{\text{TOP}}_T] - \gamma^{\text{MID}}\mathbb{E}\left[\sum_{t=2}^{T}\sum_{j=1}^{M} q^{\text{TOP}}_{t,j}\sum_{i=1}^{N} q^{\text{MID}}_{t,j,i}\|\mathbf{x}_{t,i} - \mathbf{x}_{t-1,i}\|^2\right]\\
&+ \left(\frac{64G^2}{Z} + 128D^2L^2\right)\mathbb{E}[S^{\mathbf{x}}_T] - \gamma^{\text{TOP}}\mathbb{E}\left[\sum_{t=2}^{T}\sum_{j=1}^{M} q^{\text{TOP}}_{t,j}\|\boldsymbol{q}^{\text{MID}}_{t,j} - \boldsymbol{q}^{\text{MID}}_{t-1,j}\|_1^2\right]\\
\leq{}& \mathcal{O}\left(\sqrt{(\sigma^2_{1:T} + \Sigma^2_{1:T})\log(\sigma^2_{1:T} + \Sigma^2_{1:T})}\right) + \left(\frac{64G^2}{Z} + \gamma^{\text{MID}}\right)\mathbb{E}[S^{\mathbf{x}}_{T,i^\star}]
\end{aligned}$$

$$- \frac{C_0}{2}\mathbb{E}[S_T^{\text{TOP}}] - \gamma^{\text{MID}}\mathbb{E}\left[\sum_{t=2}^{T}\sum_{j=1}^{M} q_{t,j}^{\text{TOP}} \sum_{i=1}^{N} q_{t,j,i}^{\text{MID}} \|\mathbf{x}_{t,i} - \mathbf{x}_{t-1,i}\|^2\right]$$

$$+ \left(\frac{64G^2}{Z} + 128D^2L^2\right)\mathbb{E}[S_T^{\mathbf{x}}] - \gamma^{\text{TOP}}\mathbb{E}\left[\sum_{t=2}^{T}\sum_{j=1}^{M} q_{t,j}^{\text{TOP}} \|\boldsymbol{q}_{t,j}^{\text{MID}} - \boldsymbol{q}_{t-1,j}^{\text{MID}}\|_1^2\right], \quad \text{(D.21)}$$

where the second step is due to the decomposition of $V_\star$, the third step is by Eq. (6.5) and the final step follows from Lemma 12.

Base Regret Analysis. For λ-*strongly convex* functions, we need to delve into the proof details of the base algorithm, i.e., OOMD (C.2) for strongly convex functions with step size $\eta_t = 2/(\kappa + \lambda_i t)$. For example, from Lemma 12 of Yan et al. (2023), the base regret can be bounded as

$$\text{Base-Reg} \le 4\sum_{t=2}^{T} \frac{1}{\lambda_{i^\star} t}\mathbb{E}\left[\left\|\nabla h_{t,i^\star}^{\text{sc}}(\mathbf{x}_{t,i^\star}) - \nabla h_{t-1,i^\star}^{\text{sc}}(\mathbf{x}_{t-1,i^\star})\right\|^2\right] - \frac{\kappa}{8}\mathbb{E}[S_{T,i^\star}^{\mathbf{x}}] + \mathcal{O}(1).$$

Subsequently, we analyze the empirical gradient variation defined on surrogates in each round. Denoting by $\sigma_t^2 \triangleq \max_{\mathbf{x}\in\mathcal{X}} \mathbb{E}_{f_t\sim\mathfrak{D}_t}[\|\nabla f_t(\mathbf{x}) - \nabla F_t(\mathbf{x})\|^2]$ and $\Sigma_t^2 \triangleq \mathbb{E}[\sup_{\mathbf{x}\in\mathcal{X}} \|\nabla F_t(\mathbf{x}) - \nabla F_{t-1}(\mathbf{x})\|^2]$ for simplicity,

$$\begin{aligned}
&\mathbb{E}\left[\left\|\nabla h_{t,i^\star}^{\text{sc}}(\mathbf{x}_{t,i^\star}) - \nabla h_{t-1,i^\star}^{\text{sc}}(\mathbf{x}_{t-1,i^\star})\right\|^2\right]\\
&= \mathbb{E}\left[\|\mathbf{g}_t + \lambda_{i^\star}(\mathbf{x}_{t,i^\star} - \mathbf{x}_t) - \mathbf{g}_{t-1} - \lambda_{i^\star}(\mathbf{x}_{t-1,i^\star} - \mathbf{x}_{t-1})\|^2\right]\\
&\le 2\mathbb{E}\left[\|\mathbf{g}_t - \mathbf{g}_{t-1}\|^2\right] + 2\|\lambda_{i^\star}(\mathbf{x}_{t,i^\star} - \mathbf{x}_t) + \lambda_{i^\star}(\mathbf{x}_{t-1,i^\star} - \mathbf{x}_{t-1})\|^2\\
&\le 4\left(2\sigma_t^2 + 2\sigma_{t-1}^2 + (1+2L^2)\mathbb{E}\left[\|\mathbf{x}_t - \mathbf{x}_{t-1}\|^2\right] + \mathbb{E}\left[\|\mathbf{x}_{t,i^\star} - \mathbf{x}_{t-1,i^\star}\|^2\right] + 2\Sigma_t^2\right), \quad \text{(by (6.5))}
\end{aligned}$$

where the first step is due to the property of the surrogate: $\nabla h_{t,i}^{\text{sc}}(\mathbf{x}_{t,i}) = \mathbf{g}_t + \lambda_i(\mathbf{x}_{t,i} - \mathbf{x}_t)$, and the second step is due to the Cauchy-Schwarz inequality. Plugging the above term back into the base regret and omitting the ignorable $\mathcal{O}(1)$ term, we achieve

$$\begin{aligned}
\text{Base-Reg} \le \frac{32}{\lambda_{i^\star}}\sum_{t=2}^{T}\frac{\sigma_t^2 + \sigma_{t-1}^2 + \Sigma_t^2}{t} + 16(1+2L^2)\sum_{t=2}^{T}\frac{\mathbb{E}\left[\|\mathbf{x}_t - \mathbf{x}_{t-1}\|^2\right]}{\lambda_{i^\star} t}\\
+ 16\sum_{t=2}^{T}\frac{\mathbb{E}\left[\|\mathbf{x}_{t,i^\star} - \mathbf{x}_{t-1,i^\star}\|^2\right]}{\lambda_{i^\star} t}.
\end{aligned}$$

Using Lemma 14, we control the base regret as

$$\begin{aligned}
\text{Base-Reg} &\le \mathcal{O}\left(\frac{1}{\lambda}\left(\sigma_{\max}^2 + \Sigma_{\max}^2\right)\log\frac{\sigma_{1:T}^2 + \Sigma_{1:T}^2}{\sigma_{\max}^2 + \Sigma_{\max}^2}\right)\\
&\quad + \frac{32D^2(L^2+1)}{\lambda_{i^\star}}\log\left(1 + (1+2L^2)\lambda_{i^\star}E[S_T^{\mathbf{x}}] + \lambda_{i^\star}\mathbb{E}[S_{T,i^\star}^{\mathbf{x}}]\right)\\
&\le \mathcal{O}\left(\frac{1}{\lambda}\left(\sigma_{\max}^2 + \Sigma_{\max}^2\right)\log\frac{\sigma_{1:T}^2 + \Sigma_{1:T}^2}{\sigma_{\max}^2 + \Sigma_{\max}^2}\right) + 32D^2(L^2+1)\left((1+2L^2)E[S_T^{\mathbf{x}}] + E[S_{T,i^\star}^{\mathbf{x}}]\right),
\end{aligned} \quad \text{(D.22)}$$

where the first term initializes Lemma 14 as $a_t = \sigma_t^2 + \sigma_{t-1}^2 + \Sigma_t^2$ (i.e., $a_{\max} = \mathcal{O}(\sigma_{\max}^2 + \Sigma_{\max}^2)$) and $b = 1/(\sigma_{\max}^2 + \Sigma_{\max}^2)$, the second term initializes Lemma 14 as $a_t = (1 + 2L^2)\mathbb{E}\left[\|\mathbf{x}_{t-1} - \mathbf{x}_t\|^2\right] + \mathbb{E}\left[\|\mathbf{x}_{t,i^\star} - \mathbf{x}_{t-1,i^\star}\|^2\right]$ (i.e., $a_{\max} = \left(2 + 2L^2\right) D^2$ due to Assumption 1) and $b = \lambda_{i^\star}$. The second step is due to $\log(1+x) \le x$ for $x \ge 0$.

For *α-exp-concave* functions, the base regret is bounded by (C.11). Following (6.5), we control the empirical gradient variation defined on surrogates as

$$\begin{aligned}
&\mathbb{E}\left[\sum_{t=2}^{T} \left\|\nabla h_{t,i^\star}^{\exp}(\mathbf{x}_{t,i^\star}) - \nabla h_{t-1,i^\star}^{\exp}(\mathbf{x}_{t-1,i^\star})\right\|^2\right]\\
&\le 2\mathbb{E}\left[\sum_{t=2}^{T}\|\mathbf{g}_t - \mathbf{g}_{t-1}\|^2\right] + 2\alpha_{i^\star}^2\left[\sum_{t=2}^{T}\mathbb{E}\|\mathbf{g}_t\mathbf{g}_t^\top(\mathbf{x}_{t,i^\star} - \mathbf{x}_t) - \mathbf{g}_{t-1}\mathbf{g}_{t-1}^\top(\mathbf{x}_{t-1,i^\star} - \mathbf{x}_{t-1})\|^2\right]\\
&\le 8(2\sigma_{1:T}^2 + \Sigma_{1:T}^2) + 8L^2\mathbb{E}[S_T^{\mathbf{x}}] + 4D^2\mathbb{E}\left[\sum_{t=2}^{T}\|\mathbf{g}_t\mathbf{g}_t^\top - \mathbf{g}_{t-1}\mathbf{g}_{t-1}^\top\|^2\right]\\
&\qquad + 4G^4\mathbb{E}\left[\sum_{t=2}^{T}\|(\mathbf{x}_{t,i^\star} - \mathbf{x}_t) - (\mathbf{x}_{t-1,i^\star} - \mathbf{x}_{t-1})\|^2\right] \qquad (\text{by } \alpha \in [1/T, 1])\\
&\le C_{22}(2\sigma_{1:T}^2 + \Sigma_{1:T}^2) + C_{23}\mathbb{E}[S_T^{\mathbf{x}}] + 8G^4\mathbb{E}[S_{T,i^\star}^{\mathbf{x}}],
\end{aligned}$$

where the first step uses the definition of $\nabla h_{t,i}^{\exp}(\mathbf{x}) = \mathbf{g}_t + \alpha_i\mathbf{g}_t\mathbf{g}_t^\top(\mathbf{x} - \mathbf{x}_t)$ and the last step holds by setting $C_{22} = 8 + 64D^2G^2$ and $C_{23} = 8L^2 + 64D^2G^2L^2 + 8G^4$. Then we obtain

$$\begin{aligned}
&\textsc{Base-Reg}\\
&\le \frac{16d}{\alpha_{i^\star}}\log\left(1 + \frac{\alpha_{i^\star}C_{22}}{8\kappa d}(2\sigma_{1:T}^2 + \Sigma_{1:T}^2) + \frac{\alpha_{i^\star}C_{23}}{8\kappa d}\mathbb{E}[S_T^{\mathbf{x}}] + \frac{\alpha_{i^\star}G^4}{\kappa d}\mathbb{E}[S_{T,i^\star}^{\mathbf{x}}]\right)\\
&\qquad + \frac{1}{2}\kappa D^2 - \frac{1}{4}\kappa\mathbb{E}[S_{T,i^\star}^{\mathbf{x}}] + \mathcal{O}(1)\\
&\le \frac{32d}{\alpha}\log\left(1 + \frac{\alpha C_{22}}{8\kappa d}(2\sigma_{1:T}^2 + \Sigma_{1:T}^2)\right) + \frac{2C_{23}}{\kappa}\mathbb{E}[S_T^{\mathbf{x}}]\\
&\qquad + \left(\frac{16G^4}{\kappa} - \frac{\kappa}{4}\right)\mathbb{E}[S_{T,i^\star}^{\mathbf{x}}] + \frac{1}{2}\kappa D^2 + \mathcal{O}(1),
\end{aligned} \tag{D.23}$$

where the last step is due to $\log(1+x) \le x$ for $x \ge 0$.

For *convex* functions, Lemma 21 guarantees the following:

$$\begin{aligned}
&\textsc{Base-Reg} \le 5D\sqrt{1 + \mathbb{E}[\bar{V}_{T,i^\star}^{\mathrm{c}}]} + \kappa D^2 - \frac{1}{4}\kappa\mathbb{E}[S_{T,i^\star}^{\mathbf{x}}] + \mathcal{O}(1)\\
&\le 5D\sqrt{1 + 4(2\sigma_{1:T}^2 + \Sigma_{1:T}^2) + 4L^2\mathbb{E}[S_T^{\mathbf{x}}]} + \kappa D^2 - \frac{1}{4}\kappa\mathbb{E}[S_{T,i^\star}^{\mathbf{x}}] + \mathcal{O}(1)\\
&\le 5D\sqrt{1 + 4(2\sigma_{1:T}^2 + \Sigma_{1:T}^2)} + 20DL^2\mathbb{E}[S_T^{\mathbf{x}}] + \kappa D^2 - \frac{1}{4}\kappa\mathbb{E}[S_{T,i^\star}^{\mathbf{x}}] + \mathcal{O}(1),
\end{aligned} \tag{D.24}$$

where the second step is due to Eq. (6.5).

Overall Regret Analysis. For λ-*strongly convex* functions, plugging Eq. (C.5) and Eq. (D.22) into Eq. (D.18) and letting $C_{24} = 64D^2(1+L^2)^2$, we obtain

$$\begin{aligned}
\textsc{Reg}_T &\le \mathcal{O}\left(\frac{1}{\lambda}\left(\sigma_{\max}^2 + \Sigma_{\max}^2\right)\log\frac{\sigma_{1:T}^2 + \Sigma_{1:T}^2}{\sigma_{\max}^2 + \Sigma_{\max}^2}\right) + C_{24}\mathbb{E}[S_T^{\mathbf{x}}] \\
&\quad + \left(32D^2(1+L^2) + \gamma^{\text{MID}} - \frac{1}{8}\kappa\right)\mathbb{E}[S_{T,i^\star}^{\mathbf{x}}] - \gamma^{\text{MID}}\mathbb{E}\left[\sum_{t=2}^{T}\sum_{j=1}^{M} q_{t,j}^{\text{TOP}}\sum_{i=1}^{N} q_{t,j,i}^{\text{MID}}\left\|\mathbf{x}_{t,i} - \mathbf{x}_{t-1,i}\right\|^2\right] \\
&\quad + \frac{1}{4}\kappa D^2 - \frac{C_0}{2}\mathbb{E}[S_T^{\text{TOP}}] - \gamma^{\text{TOP}}\mathbb{E}\left[\sum_{t=2}^{T}\sum_{j=1}^{M} q_{t,j}^{\text{TOP}}\|\boldsymbol{q}_{t,j}^{\text{MID}} - \boldsymbol{q}_{t-1,j}^{\text{MID}}\|_1^2\right] \\
&\le \mathcal{O}\left(\frac{1}{\lambda}\left(\sigma_{\max}^2 + \Sigma_{\max}^2\right)\log\frac{\sigma_{1:T}^2 + \Sigma_{1:T}^2}{\sigma_{\max}^2 + \Sigma_{\max}^2}\right) + \left(32D^2(1+L^2) + \gamma^{\text{MID}} - \frac{1}{8}\kappa\right)\mathbb{E}[S_{T,i^\star}^{\mathbf{x}}] \\
&\qquad + \left(4D^2C_{24} - \frac{C_0}{2}\right)\mathbb{E}[S_T^{\text{TOP}}] + (2C_{24} - \gamma^{\text{MID}})\,\mathbb{E}\left[\sum_{t=2}^{T}\sum_{j=1}^{M} q_{t,j}^{\text{TOP}}\sum_{i=1}^{N} q_{t,j,i}^{\text{MID}}\left\|\mathbf{x}_{t,i} - \mathbf{x}_{t-1,i}\right\|^2\right] \\
&\qquad\quad + \frac{1}{4}\kappa D^2 + \left(4D^2C_{24} - \gamma^{\text{TOP}}\right)\mathbb{E}\left[\sum_{t=2}^{T}\sum_{j=1}^{M} q_{t,j}^{\text{TOP}}\|\boldsymbol{q}_{t,j}^{\text{MID}} - \boldsymbol{q}_{t-1,j}^{\text{MID}}\|_1^2\right] \\
&\le \mathcal{O}\left(\frac{1}{\lambda}\left(\sigma_{\max}^2 + \Sigma_{\max}^2\right)\log\frac{\sigma_{1:T}^2 + \Sigma_{1:T}^2}{\sigma_{\max}^2 + \Sigma_{\max}^2}\right),
\end{aligned}$$

where the second step follows from Lemma 5 and the last step requires $\gamma^{\text{TOP}} \ge 4D^2C_{24}$, $\gamma^{\text{MID}} \ge 2C_{24}$, $\kappa \ge 256D^2(1+L^2) + 8\gamma^{\text{MID}}$, and $C_0 \ge 8D^2C_{24}$. For α-*exp-concave* functions, plugging Eq. (C.6) and Eq. (D.23) into Eq. (D.19), we obtain

$$\begin{aligned}
\textsc{Reg}_T &\le \mathcal{O}\left(\frac{d}{\alpha}\log\left(\sigma_{1:T}^2 + \Sigma_{1:T}^2\right)\right) + \frac{2C_{23}}{\kappa}\mathbb{E}[S_T^{\mathbf{x}}] + \left(\frac{16G^4}{\kappa} + \gamma^{\text{MID}} - \frac{\kappa}{4}\right)\mathbb{E}[S_{T,i^\star}^{\mathbf{x}}] \\
&\quad + \frac{1}{2}\kappa D^2 - \frac{C_0}{2}\mathbb{E}[S_T^{\text{TOP}}] - \gamma^{\text{MID}}\mathbb{E}\left[\sum_{t=2}^{T}\sum_{j=1}^{M} q_{t,j}^{\text{TOP}}\sum_{i=1}^{N} q_{t,j,i}^{\text{MID}}\left\|\mathbf{x}_{t,i} - \mathbf{x}_{t-1,i}\right\|^2\right] \\
&\quad - \gamma^{\text{TOP}}\mathbb{E}\left[\sum_{t=2}^{T}\sum_{j=1}^{M} q_{t,j}^{\text{TOP}}\|\boldsymbol{q}_{t,j}^{\text{MID}} - \boldsymbol{q}_{t-1,j}^{\text{MID}}\|_1^2\right] \\
&\le \mathcal{O}\left(\frac{d}{\alpha}\log\left(\sigma_{1:T}^2 + \Sigma_{1:T}^2\right)\right) + \left(\frac{16G^4}{\kappa} + \gamma^{\text{MID}} - \frac{\kappa}{4}\right)\mathbb{E}[S_{T,i^\star}^{\mathbf{x}}] + \frac{1}{2}\kappa D^2 \\
&\quad + \left(\frac{8D^2C_{23}}{\kappa} - \frac{C_0}{2}\right)\mathbb{E}[S_T^{\text{TOP}}] + \left(\frac{4C_{23}}{\kappa} - \gamma^{\text{MID}}\right)\mathbb{E}\left[\sum_{t=2}^{T}\sum_{j=1}^{M} q_{t,j}^{\text{TOP}}\sum_{i=1}^{N} q_{t,j,i}^{\text{MID}}\left\|\mathbf{x}_{t,i} - \mathbf{x}_{t-1,i}\right\|^2\right] \\
&\quad + \left(\frac{8D^2C_{23}}{\kappa} - \gamma^{\text{TOP}}\right)\mathbb{E}\left[\sum_{t=2}^{T}\sum_{j=1}^{M} q_{t,j}^{\text{TOP}}\|\boldsymbol{q}_{t,j}^{\text{MID}} - \boldsymbol{q}_{t-1,j}^{\text{MID}}\|_1^2\right] \le \mathcal{O}\left(\frac{d}{\alpha}\log\left(\sigma_{1:T}^2 + \Sigma_{1:T}^2\right)\right),
\end{aligned}$$

where the second step follows from Lemma 5 and the last step requires $\gamma^{\text{TOP}} \ge 8D^2C_{23}$, $\gamma^{\text{MID}} \ge 4C_{23}$, $\kappa \ge 64G^4 + 4\gamma^{\text{MID}}$, and $C_0 \ge 16D^2C_{23}$. For *convex* functions, plugging

Eq. (D.21) and Eq. (D.24) into Eq. (D.20), we obtain

$$
\begin{aligned}
&\textsc{Reg}_T \leq \mathcal{O}\left(\sqrt{(\sigma_{1:T}^2 + \Sigma_{1:T}^2)\log\left(\sigma_{1:T}^2 + \Sigma_{1:T}^2\right)}\right) + \left(\frac{64G^2}{Z} + \gamma^{\text{MID}} - \frac{\kappa}{4}\right)\mathbb{E}[S_{T,i^\star}^{\mathbf{x}}] \\
&\quad + C_{25}\mathbb{E}[S_T^{\mathbf{x}}] - \frac{C_0}{2}\mathbb{E}[S_T^{\text{TOP}}] - \gamma^{\text{MID}}\mathbb{E}\left[\sum_{t=2}^T\sum_{j=1}^M q_{t,j}^{\text{TOP}}\sum_{i=1}^N q_{t,j,i}^{\text{MID}}\left\|\mathbf{x}_{t,i} - \mathbf{x}_{t-1,i}\right\|^2\right] + \kappa D^2 \\
&\quad - \gamma^{\text{TOP}}\mathbb{E}\left[\sum_{t=2}^T\sum_{j=1}^M q_{t,j}^{\text{TOP}}\|\boldsymbol{q}_{t,j}^{\text{MID}} - \boldsymbol{q}_{t-1,j}^{\text{MID}}\|_1^2\right] \\
&\leq \mathcal{O}\left(\sqrt{(\sigma_{1:T}^2 + \Sigma_{1:T}^2)\log\left(\sigma_{1:T}^2 + \Sigma_{1:T}^2\right)}\right) + \left(\frac{64G^2}{Z} + \gamma^{\text{MID}} - \frac{\kappa}{4}\right)\mathbb{E}[S_{T,i^\star}^{\mathbf{x}}] \\
&\quad + \left(4D^2C_{25} - \frac{C_0}{2}\right)\mathbb{E}[S_T^{\text{TOP}}] + \kappa D^2 + \left(4D^2C_{25} - \gamma^{\text{TOP}}\right)\mathbb{E}\left[\sum_{t=2}^T\sum_{j=1}^M q_{t,j}^{\text{TOP}}\|\boldsymbol{q}_{t,j}^{\text{MID}} - \boldsymbol{q}_{t-1,j}^{\text{MID}}\|_1^2\right] \\
&\quad + (2C_{25} - \gamma^{\text{MID}})\,\mathbb{E}\left[\sum_{t=2}^T\sum_{j=1}^M q_{t,j}^{\text{TOP}}\sum_{i=1}^N q_{t,j,i}^{\text{MID}}\left\|\mathbf{x}_{t,i} - \mathbf{x}_{t-1,i}\right\|^2\right] \\
&\leq \mathcal{O}\left(\sqrt{(\sigma_{1:T}^2 + \Sigma_{1:T}^2)\log\left(\sigma_{1:T}^2 + \Sigma_{1:T}^2\right)}\right),
\end{aligned}
$$

where the first step is by setting $C_{25} = 20DL^2 + \frac{64G^2}{Z} + 128D^2L^2$, the second step follows from Lemma 5 and the last step requires $\gamma^{\text{TOP}} \geq 4D^2C_{25}$, $\gamma^{\text{MID}} \geq 2C_{25}$, $\kappa \geq 256G^2 + 4\gamma^{\text{MID}}$, and $C_0 \geq 8D^2C_{25}$.

At last, we determine the specific values of C_0, γ^{TOP}, and γ^{MID}. These parameters need to satisfy the following requirements:

$$
\begin{gathered}
C_0 \geq 1,\ C_0 \geq 8D,\ C_0 \geq 4\gamma^{\text{TOP}},\ C_0 \geq 8D^2C_{24},\ C_0 \geq 8D^2C_{23},\ C_0 \geq 8D^2C_{25}, \\
\gamma^{\text{TOP}} \geq 4D^2C_{24},\ \gamma^{\text{TOP}} \geq 8D^2C_{23},\ \gamma^{\text{TOP}} \geq 4D^2C_{25},\ \gamma^{\text{MID}} \geq 2C_{24}, \\
\gamma^{\text{MID}} \geq 4C_{23},\ \text{and } \gamma^{\text{MID}} \geq 2C_{25}.
\end{gathered}
$$

As a result, we set

$$
\begin{aligned}
C_0 &= \max\left\{1, 8D, 4\gamma^{\text{TOP}}, 8D^2C_{24}, 8D^2C_{23}, 8D^2C_{25}\right\}, \\
\gamma^{\text{TOP}} &= \max\left\{4D^2C_{24}, 8D^2C_{23}, 4D^2\left(20DL^2 + 64G^2 + 128D^2L^2\right)\right\}, \\
\gamma^{\text{MID}} &= \max\left\{2C_{24}, 4C_{23}, 20DL^2 + 64G^2 + 128D^2L^2\right\},
\end{aligned}
$$

where $Z = \max\{GD + \gamma^{\text{MID}}D^2, 1 + \gamma^{\text{MID}}D^2 + 2\gamma^{\text{TOP}}\}$, $C_{23} = 8L^2 + 64D^2G^2L^2 + 8G^4$, $C_{24} = 64D^2(1 + L^2)^2$, and $C_{25} = 20DL^2 + \frac{64G^2}{Z} + 128D^2L^2$. The proof is finished. ∎

D.4 Proof of Theorem 6

Proof Recall that we denote by $\mathbf{g}_t = \nabla f_t(\mathbf{x}_t)$ for simplicity. To begin with, we provide a proof for Eq. (6.6):

$$\begin{aligned}\mathbb{E}[\bar{V}_T] \leq{} & 5\mathbb{E}\left[\sum_{t=2}^{T} \|\nabla f_t(\mathbf{x}_t) - \nabla F_t(\mathbf{x}_t)\|^2\right] + 5\mathbb{E}\left[\sum_{t=2}^{T} \|\nabla F_t(\mathbf{x}_t) - \nabla F_t(\mathbf{x}^\star)\|^2\right] \\ & +5\mathbb{E}\left[\sum_{t=2}^{T} \|\nabla F_t(\mathbf{x}^\star) - \nabla F_{t-1}(\mathbf{x}^\star)\|^2\right] + 5\mathbb{E}\left[\sum_{t=2}^{T} \|\nabla F_{t-1}(\mathbf{x}^\star) - \nabla F_{t-1}(\mathbf{x}_{t-1})\|^2\right] \\ & +5\mathbb{E}\left[\sum_{t=2}^{T} \|\nabla F_{t-1}(\mathbf{x}_{t-1}) - \nabla f_{t-1}(\mathbf{x}_{t-1})\|^2\right] \leq 10\sigma_{1:T}^2 + 5\Sigma_{1:T}^2 + 20L\mathbb{E}\left[\sum_{t=1}^{T} \mathcal{D}_{F_t}(\mathbf{x}^\star, \mathbf{x}_t)\right],\end{aligned} \tag{D.25}$$

where the first step is due to Cauchy-Schwarz inequality and the last step is because of the definitions of $\sigma_{1:T}^2$ and $\Sigma_{1:T}^2$ (given in Section 6.2) and the analysis proposed in Section 5.3.

In the following, we first give regret decompositions for different curvature types, then we analyze the meta and base regret, and combine them for the final regret guarantees.

Regret Decomposition. For λ-*strongly convex* functions, similar to the decomposition in Appendix C.4, we have

$$\begin{aligned}\mathbb{E}[\textsc{Reg}_T] ={} & \mathbb{E}\left[\sum_{t=1}^{T} \langle \nabla F_t(\mathbf{x}_t), \mathbf{x}_t - \mathbf{x}^\star\rangle\right] - \frac{1}{2}\mathbb{E}\left[\sum_{t=1}^{T} \mathcal{D}_{F_t}(\mathbf{x}^\star, \mathbf{x}_t)\right] - \frac{1}{2}\mathbb{E}\left[\sum_{t=1}^{T} \mathcal{D}_{F_t}(\mathbf{x}^\star, \mathbf{x}_t)\right] \\ \leq{} & \mathbb{E}\left[\sum_{t=1}^{T} \langle \mathbf{g}_t, \mathbf{x}_t - \mathbf{x}^\star\rangle\right] - \frac{\lambda}{4}\mathbb{E}\left[\sum_{t=1}^{T} \|\mathbf{x}_t - \mathbf{x}^\star\|^2\right] - \frac{1}{2}\mathbb{E}\left[\sum_{t=1}^{T} \mathcal{D}_{F_t}(\mathbf{x}^\star, \mathbf{x}_t)\right] \\ \leq{} & \underbrace{\mathbb{E}\left[\sum_{t=1}^{T} \langle \mathbf{g}_t, \mathbf{x}_t - \mathbf{x}_{t,i^\star}\rangle\right] - \frac{\lambda_{i^\star}}{4}\mathbb{E}\left[\sum_{t=1}^{T} \|\mathbf{x}_t - \mathbf{x}_{t,i^\star}\|^2\right]}_{\textsc{Meta-Reg}} \\ & + \underbrace{\mathbb{E}\left[\sum_{t=1}^{T} h_{t,i^\star}^{\text{sc}}(\mathbf{x}_{t,i^\star}) - h_{t,i^\star}^{\text{sc}}(\mathbf{x}^\star)\right]}_{\textsc{Base-Reg}} - \frac{1}{2}\mathbb{E}\left[\sum_{t=1}^{T} \mathcal{D}_{F_t}(\mathbf{x}^\star, \mathbf{x}_t)\right],\end{aligned}$$

where the first and second steps rely on the expected loss function $F_t(\mathbf{x}) = \mathbb{E}[f_t(\mathbf{x})]$; in particular, the second step additionally requires that $F_t(\cdot)$ be *strongly convex*. The third step follows from the definition of the surrogate function $h_{t,i}^{\text{sc}}(\mathbf{x}) = \langle \mathbf{g}_t, \mathbf{x}\rangle + \frac{\lambda_i}{4}\|\mathbf{x} - \mathbf{x}_t\|^2$, where $\lambda_i \in \mathcal{H}$ is defined in (2.5).

For α-*exp-concave* functions, following the similar decomposition as in the proof of Theorem 4 in Appendix C.4, we decompose the regret as

$$\begin{aligned}\mathbb{E}[\textsc{Reg}_T] ={} & \mathbb{E}\left[\sum_{t=1}^{T} \langle \nabla F_t(\mathbf{x}_t), \mathbf{x}_t - \mathbf{x}^\star\rangle\right] - \frac{1}{2}\mathbb{E}\left[\sum_{t=1}^{T} \mathcal{D}_{F_t}(\mathbf{x}^\star, \mathbf{x}_t)\right] - \frac{1}{2}\mathbb{E}\left[\sum_{t=1}^{T} \mathcal{D}_{F_t}(\mathbf{x}^\star, \mathbf{x}_t)\right] \\ \leq{} & \mathbb{E}\left[\sum_{t=1}^{T} \langle \mathbf{g}_t, \mathbf{x}_t - \mathbf{x}^\star\rangle\right] - \frac{\alpha}{4}\mathbb{E}\left[\sum_{t=1}^{T} \langle \mathbf{g}_t, \mathbf{x}_t - \mathbf{x}^\star\rangle^2\right] - \frac{1}{2}\mathbb{E}\left[\sum_{t=1}^{T} \mathcal{D}_{F_t}(\mathbf{x}^\star, \mathbf{x}_t)\right]\end{aligned}$$

$$\leq \underbrace{\mathbb{E}\left[\sum_{t=1}^{T}\langle \mathbf{g}_t, \mathbf{x}_t - \mathbf{x}_{t,i^\star}\rangle\right] - \frac{\alpha_{i^\star}}{4}\mathbb{E}\left[\sum_{t=1}^{T}\langle \mathbf{g}_t, \mathbf{x}_t - \mathbf{x}_{t,i^\star}\rangle^2\right]}_{\text{Meta-Reg}}$$
$$+ \underbrace{\mathbb{E}\left[\sum_{t=1}^{T} h_{t,i^\star}^{\exp}(\mathbf{x}_{t,i^\star}) - h_{t,i^\star}^{\exp}(\mathbf{x}^\star)\right]}_{\text{Base-Reg}} - \frac{1}{2}\mathbb{E}\left[\sum_{t=1}^{T}\mathcal{D}_{F_t}(\mathbf{x}^\star, \mathbf{x}_t)\right],$$

where the first and second steps rely on the expected loss function $F_t(\mathbf{x}) = \mathbb{E}[f_t(\mathbf{x})]$; in particular, the second step additionally requires that $f_t(\cdot)$ be *exp-concave*. The third step follows from the definition of the surrogate function $h_{t,i}^{\exp}(\mathbf{x}) = \langle \mathbf{g}_t, \mathbf{x}\rangle + \frac{\alpha_i}{4}\langle \mathbf{g}_t, \mathbf{x} - \mathbf{x}_t\rangle^2$, where $\alpha_i \in \mathcal{H}$ is defined in (2.5).

For *convex* functions, we decompose the regret as

$$\mathbb{E}[\text{Reg}_T] = \mathbb{E}\left[\sum_{t=1}^{T} F_t(\mathbf{x}_t) - \sum_{t=1}^{T} F_t(\mathbf{x}^\star)\right] = \mathbb{E}\left[\sum_{t=1}^{T}\langle \nabla F_t(\mathbf{x}_t), \mathbf{x}_t - \mathbf{x}^\star\rangle\right] - \mathbb{E}\left[\sum_{t=1}^{T}\mathcal{D}_{F_t}(\mathbf{x}^\star, \mathbf{x}_t)\right]$$
$$= \mathbb{E}\left[\sum_{t=1}^{T}\langle \mathbf{g}_t, \mathbf{x}_t - \mathbf{x}^\star\rangle\right] - \mathbb{E}\left[\sum_{t=1}^{T}\mathcal{D}_{F_t}(\mathbf{x}^\star, \mathbf{x}_t)\right]$$
$$= \underbrace{\mathbb{E}\left[\sum_{t=1}^{T}\langle \mathbf{g}_t, \mathbf{x}_t - \mathbf{x}_{t,i^\star}\rangle\right]}_{\text{Meta-Reg}} + \underbrace{\mathbb{E}\left[\sum_{t=1}^{T} h_{t,i^\star}^{\mathrm{c}}(\mathbf{x}_{t,i^\star}) - h_{t,i^\star}^{\mathrm{c}}(\mathbf{x}^\star)\right]}_{\text{Base-Reg}} - \mathbb{E}\left[\sum_{t=1}^{T}\mathcal{D}_{F_t}(\mathbf{x}^\star, \mathbf{x}_t)\right],$$

where the first and third step use $F_t(\mathbf{x}) = \mathbb{E}[f_t(\mathbf{x})]$, the second step uses the definition of Bregman divergence, and the fourth step is due to $h_{t,i}^{\mathrm{c}}(\mathbf{x}) = \langle \mathbf{g}_t, \mathbf{x}\rangle$.

Meta Regret Analysis. Our Algorithm 5 can be applied to the SEA model without any algorithm modifications. As a result, we directly use the same parameter configurations as in the proof of Theorem 4 (i.e., in Appendix C.4).

For *strongly convex* and *exp-concave* functions, the meta regret is bounded in a similar way as (B.8) and (B.9), and thus omitted here.

For *convex* functions, the meta regret can be bounded as

$$\text{Meta-Reg} \leq \mathbb{E}\left[C_0\sqrt{4G^2D^2 + D^2\bar{V}_T} + 2GDC_2\right] \leq C_0\sqrt{4G^2D^2 + D^2\mathbb{E}[\bar{V}_T]} + 2GDC_2$$
$$\leq C_0\sqrt{4G^2D^2 + 5D^2(2\sigma_{1:T}^2 + \Sigma_{1:T}^2) + 20D^2L\mathbb{E}\left[\sum_{t=1}^{T}\mathcal{D}_{F_t}(\mathbf{x}^\star, \mathbf{x}_t)\right]} + 2GDC_2 \qquad \text{(by (6.6))}$$
$$\leq \mathcal{O}\left(\sqrt{\sigma_{1:T}^2 + \Sigma_{1:T}^2}\right) + \mathcal{O}(C_{26}) + \frac{C_0}{2C_{26}}\mathbb{E}\left[\sum_{t=1}^{T}\mathcal{D}_{F_t}(\mathbf{x}^\star, \mathbf{x}_t)\right],$$

where the second step is by Jensen's inequality and the last step is due to AM-GM inequality (Lemma 18). C_{26} is a constant to be specified.

Base Regret Analysis. For λ-*strongly convex* functions, similar to the analysis in Appendix D.3, the base regret can be bounded as

$$\text{Base-Reg} \le 4\sum_{t=2}^{T}\frac{1}{\lambda_{i^\star} t}\mathbb{E}\left[\left\|\nabla h^{\text{sc}}_{t,i^\star}(\mathbf{x}_{t,i^\star}) - \nabla h^{\text{sc}}_{t-1,i^\star}(\mathbf{x}_{t-1,i^\star})\right\|^2\right] + \mathcal{O}(1).$$

Subsequently, we analyze the empirical gradient variation defined on surrogates in each round, i.e., $\|\nabla h^{\text{sc}}_{t,i^\star}(\mathbf{x}_{t,i^\star}) - \nabla h^{\text{sc}}_{t-1,i^\star}(\mathbf{x}_{t-1,i^\star})\|^2$. Denoting by $\sigma_t^2 \triangleq \max_{\mathbf{x}\in\mathcal{X}} \mathbb{E}_{f_t\sim\mathfrak{D}_t}[\|\nabla f_t(\mathbf{x}) - \nabla F_t(\mathbf{x})\|^2]$ and $\Sigma_t^2 \triangleq \mathbb{E}[\sup_{\mathbf{x}\in\mathcal{X}}\|\nabla F_t(\mathbf{x}) - \nabla F_{t-1}(\mathbf{x})\|^2]$ for simplicity,

$$\begin{aligned}
&\mathbb{E}\left[\left\|\nabla h^{\text{sc}}_{t,i^\star}(\mathbf{x}_{t,i^\star}) - \nabla h^{\text{sc}}_{t-1,i^\star}(\mathbf{x}_{t-1,i^\star})\right\|^2\right]\\
&= \mathbb{E}\left[\left\|\mathbf{g}_t + \frac{\lambda_{i^\star}}{2}(\mathbf{x}_{t,i^\star} - \mathbf{x}_t) - \mathbf{g}_{t-1} - \frac{\lambda_{i^\star}}{2}(\mathbf{x}_{t-1,i^\star} - \mathbf{x}_{t-1})\right\|^2\right]\\
&\le 3\mathbb{E}\left[\|\mathbf{g}_t - \mathbf{g}_{t-1}\|^2\right] + 3\left\|\frac{\lambda_{i^\star}}{2}(\mathbf{x}_{t,i^\star} - \mathbf{x}_t)\right\|^2 + 3\left\|\frac{\lambda_{i^\star}}{2}(\mathbf{x}_{t-1,i^\star} - \mathbf{x}_{t-1})\right\|^2\\
&\le 15(\sigma_t^2 + \sigma_{t-1}^2 + 2L\mathbb{E}\left[\mathcal{D}_{F_t}(\mathbf{x}^\star, \mathbf{x}_t)\right] + 2L\left[\mathcal{D}_{F_{t-1}}(\mathbf{x}^\star, \mathbf{x}_{t-1})\right] + \Sigma_t^2) \qquad \text{(by (6.6))}\\
&\qquad + \lambda_{i^\star}^2\mathbb{E}\left[\|\mathbf{x}_{t,i^\star} - \mathbf{x}_t\|^2\right] + \lambda_{i^\star}^2\mathbb{E}\left[\|\mathbf{x}_{t-1,i^\star} - \mathbf{x}_{t-1}\|^2\right],
\end{aligned}$$

where the first step is due to the property of the surrogate: $\nabla h^{\text{sc}}_{t,i}(\mathbf{x}_{t,i}) = \mathbf{g}_t + \frac{\lambda_i}{2}(\mathbf{x}_{t,i} - \mathbf{x}_t)$, and the second step is due to the Cauchy-Schwarz inequality. Plugging the above term back into the base regret and omitting the ignorable $\mathcal{O}(1)$ term, we achieve

$$\begin{aligned}
\text{Base-Reg} \le\ & \frac{60}{\lambda_{i^\star}}\sum_{t=2}^{T}\frac{\sigma_t^2 + \sigma_{t-1}^2 + \Sigma_t^2}{t} + 120L\sum_{t=2}^{T}\frac{\mathbb{E}\left[\mathcal{D}_{F_t}(\mathbf{x}^\star, \mathbf{x}_t) + \mathcal{D}_{F_{t-1}}(\mathbf{x}^\star, \mathbf{x}_{t-1})\right]}{\lambda_{i^\star} t}\\
&+ 4\sum_{t=2}^{T}\frac{\lambda_{i^\star}^2\mathbb{E}\left[\|\mathbf{x}_{t,i^\star} - \mathbf{x}_t\|^2\right] + \lambda_{i^\star}^2\mathbb{E}\left[\|\mathbf{x}_{t-1,i^\star} - \mathbf{x}_{t-1}\|^2\right]}{\lambda_{i^\star} t},
\end{aligned}$$

Using Lemma 14, we control the base regret as

$$\begin{aligned}
\text{Base-Reg} &\le \mathcal{O}\left(\frac{1}{\lambda}\left(\sigma_{\max}^2 + \Sigma_{\max}^2\right)\log\frac{\sigma_{1:T}^2 + \Sigma_{1:T}^2}{\sigma_{\max}^2 + \Sigma_{\max}^2}\right)\\
&\quad + \frac{480LGD}{\lambda_{i^\star}}\log\left(1 + 2\lambda_{i^\star}\mathbb{E}\left[\sum_{t=1}^{T}\mathcal{D}_{F_t}(\mathbf{x}^\star, \mathbf{x}_t)\right]\right) + \frac{8D^2}{\lambda_{i^\star}}\log\left(1 + 2\lambda_{i^\star}^3\mathbb{E}\left[\sum_{t=1}^{T}\|\mathbf{x}_{t,i^\star} - \mathbf{x}_t\|^2\right]\right)\\
&\le \mathcal{O}\left(\frac{1}{\lambda}\left(\sigma_{\max}^2 + \Sigma_{\max}^2\right)\log\frac{\sigma_{1:T}^2 + \Sigma_{1:T}^2}{\sigma_{\max}^2 + \Sigma_{\max}^2}\right) + \mathcal{O}(\log C_{27} + \log C_{28})\\
&\quad + \frac{960LGD}{C_{27}}\mathbb{E}\left[\sum_{t=1}^{T}\mathcal{D}_{F_t}(\mathbf{x}^\star, \mathbf{x}_t)\right] + \frac{16D^2}{C_{28}}\mathbb{E}\left[\sum_{t=2}^{T}\|\mathbf{x}_{t,i^\star} - \mathbf{x}_t\|^2\right],
\end{aligned}$$

where the first term initializes Lemma 14 as $a_t = \sigma_t^2 + \sigma_{t-1}^2 + \Sigma_t^2$ (i.e., $a_{\max} = \mathcal{O}(\sigma_{\max}^2 + \Sigma_{\max}^2)$) and $b = 1/(\sigma_{\max}^2 + \Sigma_{\max}^2)$, the second term initializes Lemma 14 as $a_t = \mathbb{E}\left[\mathcal{D}_{F_t}(\mathbf{x}^\star, \mathbf{x}_t)\right] + \mathbb{E}\left[\mathcal{D}_{F_{t-1}}(\mathbf{x}^\star, \mathbf{x}_{t-1})\right]$ (i.e., $a_{\max} = 4GD$ due to Assumption 1 and Assumption 2) and $b = \lambda_{i^\star}$, the third term initializes Lemma 14 as $a_t = \mathbb{E}\left[\lambda_{i^\star}^2 \|\mathbf{x}_{t,i^\star} - \mathbf{x}_t\|^2 + \lambda_{i^\star}^2 \|\mathbf{x}_{t-1,i^\star} - \mathbf{x}_{t-1}\|^2\right]$ (i.e., $a_{\max} = 2D^2$ due to $\lambda_i \leq 1$ and Assumption 1) and $b = \lambda_{i^\star}$. The $\mathcal{O}(1)$ term contains ignorable terms like $\mathcal{O}(1/\lambda)$. The second step requires $C_{27}, C_{28} \geq 1$ by Lemma 15.

For α-*exp-concave* functions, the base regret is bounded by (C.11). Following (6.6), we control the empirical gradient variation defined on surrogates as

$$\mathbb{E}\left[\sum_{t=2}^T \left\|\nabla h_{t,i^\star}^{\exp}(\mathbf{x}_{t,i^\star}) - \nabla h_{t-1,i^\star}^{\exp}(\mathbf{x}_{t-1,i^\star})\right\|^2\right] \leq 3\mathbb{E}[\bar{V}_T] + 6\sum_{t=1}^T \left\|\frac{\alpha_{i^\star}}{2}\mathbf{g}_t \langle \mathbf{g}_t, \mathbf{x}_t - \mathbf{x}_{t,i^\star}\rangle\right\|^2$$
$$\leq 15(2\sigma_{1:T}^2 + \Sigma_{1:T}^2) + 60L\mathbb{E}\left[\sum_{t=1}^T \mathcal{D}_{F_t}(\mathbf{x}^\star, \mathbf{x}_t)\right] + 2\alpha_{i^\star}^2 G^2 \mathbb{E}\left[\sum_{t=1}^T \langle \mathbf{g}_t, \mathbf{x}_t - \mathbf{x}_{t,i^\star}\rangle\right]^2.$$

Plugging the surrogate's empirical gradient variation back to the base regret, we obtain

$$\text{BASE-REG} \leq \frac{16d}{\alpha_{i^\star}} \log\left(1 + \frac{15\alpha_{i^\star}}{8d}(2\sigma_{1:T}^2 + \Sigma_{1:T}^2) + \frac{15L\alpha_{i^\star}}{2d}\mathbb{E}\left[\sum_{t=1}^T \mathcal{D}_{F_t}(\mathbf{x}^\star, \mathbf{x}_t)\right]\right.$$
$$\left. + \frac{\alpha_{i^\star}^3 G^2}{4d}\mathbb{E}\left[\sum_{t=1}^T \langle \mathbf{g}_t, \mathbf{x}_t - \mathbf{x}_{t,i^\star}\rangle^2\right]\right) \leq \mathcal{O}\left(\frac{d}{\alpha}\log\left(\sigma_{1:T}^2 + \Sigma_{1:T}^2\right)\right) + \mathcal{O}(\log C_{29})$$
$$+ \frac{120L}{C_{29}}\mathbb{E}\left[\sum_{t=1}^T \mathcal{D}_{F_t}(\mathbf{x}^\star, \mathbf{x}_t)\right] + \frac{4G^2}{C_{29}}\mathbb{E}\left[\sum_{t=1}^T \langle \mathbf{g}_t, \mathbf{x}_t - \mathbf{x}_{t,i^\star}\rangle^2\right],$$

where the second step requires $C_{29} \geq 1$ by Lemma 15.

For *convex* functions, the base regret can be bounded as

$$\text{BASE-REG} \leq 5D\sqrt{1 + \mathbb{E}[\bar{V}_T]} \leq 5D\sqrt{1 + 10\sigma_{1:T}^2 + 5\Sigma_{1:T}^2 + 20L\mathbb{E}\left[\sum_{t=1}^T \mathcal{D}_{F_t}(\mathbf{x}^\star, \mathbf{x}_t)\right]}$$
$$\leq \mathcal{O}\left(\sqrt{\sigma_{1:T}^2 + \Sigma_{1:T}^2}\right) + \mathcal{O}(C_{30}) + \frac{5D}{2C_{30}}\mathbb{E}\left[\sum_{t=1}^T \mathcal{D}_{F_t}(\mathbf{x}^\star, \mathbf{x}_t)\right],$$

where the first step is by Jensen's inequality, the second step is due to (6.6), and the last step is because of AM-GM inequality (Lemma 18). C_{30} is a constant to be specified.

Overall Regret Analysis. For λ-*strongly convex* functions, by combining the meta and base regret, it holds that

$$\text{REG}_T \leq \mathcal{O}\left(\frac{1}{\lambda}\left(\sigma_{\max}^2 + \Sigma_{\max}^2\right)\log\frac{\sigma_{1:T}^2 + \Sigma_{1:T}^2}{\sigma_{\max}^2 + \Sigma_{\max}^2}\right) + \mathcal{O}(C_3 + \log C_{27} + \log C_{28})$$
$$+ \left(\frac{C_0 D^2}{2C_3} + \frac{16D^2}{C_{28}} - \frac{\lambda_{i^\star}}{4}\right)\mathbb{E}\left[\sum_{t=1}^T \|\mathbf{x}_t - \mathbf{x}_{t,i^\star}\|^2\right] + \left(\frac{960LGD}{C_{27}} - \frac{1}{2}\right)\mathbb{E}\left[\sum_{t=1}^T \mathcal{D}_{F_t}(\mathbf{x}^\star, \mathbf{x}_t)\right]$$
$$\leq \mathcal{O}\left(\frac{1}{\lambda}\left(\sigma_{\max}^2 + \Sigma_{\max}^2\right)\log\frac{\sigma_{1:T}^2 + \Sigma_{1:T}^2}{\sigma_{\max}^2 + \Sigma_{\max}^2}\right),$$

by choosing $C_{27} = \max\{1, 1920LGD\}$, $C_{28} = \max\{1, 128D^2/\lambda_{i^\star}\}$ and $C_3 = 4C_0 D^2/\lambda_{i^\star}$. Note that such a parameter configuration will only add an $\mathcal{O}(1/\lambda)$ factor to the final bound, which can be absorbed.

For α-*exp-concave* functions, by combining the meta and base regret, it holds that

$$\begin{aligned}\textsc{Reg}_T \leq{}& \mathcal{O}\left(\frac{d}{\alpha}\log\left(\sigma_{1:T}^2 + \Sigma_{1:T}^2\right)\right) + \mathcal{O}(C_4 + \log C_{29}) + \left(\frac{120L}{C_{29}} - \frac{1}{2}\right)\mathbb{E}\left[\sum_{t=1}^T \mathcal{D}_{F_t}(\mathbf{x}^\star, \mathbf{x}_t)\right] \\ &+ \left(\frac{C_0}{2C_4} + \frac{4G^2}{C_{29}} - \frac{\alpha_{i^\star}}{4}\right)\mathbb{E}\left[\sum_{t=1}^T \langle \mathbf{g}_t, \mathbf{x}_t - \mathbf{x}_{t,i^\star}\rangle^2\right] \leq \mathcal{O}\left(\frac{d}{\alpha}\log\left(\sigma_{1:T}^2 + \Sigma_{1:T}^2\right)\right),\end{aligned}$$

by choosing $C_{29} = \max\{1, 240L, 32G^2/\alpha_{i^\star}\}$ and $C_4 = 4C_0/\alpha_{i^\star}$. Note that such a parameter configuration will only add an $\mathcal{O}(1/\alpha)$ factor to the final regret bound, which is absorbed.

For *convex* functions, by combining the meta and base regret, it holds that

$$\begin{aligned}\textsc{Reg}_T \leq{}& \mathcal{O}\left(\sqrt{\sigma_{1:T}^2 + \Sigma_{1:T}^2}\right) + \mathcal{O}(C_{26} + C_{30}) + \left(\frac{C_0}{2C_{26}} + \frac{5D}{2C_{30}} - 1\right)\mathbb{E}\left[\sum_{t=1}^T \mathcal{D}_{F_t}(\mathbf{x}^\star, \mathbf{x}_t)\right] \\ \leq{}& \mathcal{O}\left(\sqrt{\sigma_{1:T}^2 + \Sigma_{1:T}^2}\right),\end{aligned}$$

by choosing $C_{26} = C_0$ and $C_{30} = 5D$.

Note that the constants $C_3, C_4, C_{26}, C_{27}, C_{28}, C_{29}, C_{30}$ only exist in analysis and and hence our choices of them are feasible.

■

D.5 Proof of Theorem 7

Proof For the dishonest case, the two-player game degenerates to two separate online convex optimization problems. Therefore, the results for both bilinear and strongly-convex-strongly-concave games follow directly from Appendix C.3.

For the honest case, we focus on the convex-concave game, where $f(\mathbf{x}, \mathbf{y})$ is convex in $\mathbf{x}$ and concave in $\mathbf{y}$, which subsumes both bilinear and strongly-convex-strongly-concave games. The proof begins by analyzing the regret of player-$\mathbf{x}$, following the structure in Appendix C.3. Leveraging the benign structure of the game, we upper bound the empirical gradient variation by $S_T^{\mathbf{x}} \triangleq \sum_{t=2}^T \|\mathbf{x}_t - \mathbf{x}_{t-1}\|^2$ and $S_T^{\mathbf{y}} \triangleq \sum_{t=2}^T \|\mathbf{y}_t - \mathbf{y}_{t-1}\|^2$. By symmetry, we derive a corresponding bound for player-$\mathbf{y}$, and combine the two via cancellation to obtain the final result.

To start, we denote $\gamma^{\text{MID}}, \gamma^{\text{TOP}}, S_T^{\text{TOP}}, q_{t,j}^{\text{TOP}}, Z$ for player-$\mathbf{x}$ and player-$\mathbf{y}$ by $\gamma_{\mathbf{x}}^{\text{MID}}, \gamma_{\mathbf{x}}^{\text{TOP}}$, $S_T^{\text{TOP},\mathbf{x}}, q_{t,j}^{\text{TOP},\mathbf{x}}, Z^{\mathbf{x}}$ and $\gamma_{\mathbf{y}}^{\text{MID}}, \gamma_{\mathbf{y}}^{\text{TOP}}, S_T^{\text{TOP},\mathbf{y}}, q_{t,j}^{\text{TOP},\mathbf{y}}, Z^{\mathbf{y}}$ separately. We choose $D = \sqrt{2}$. For the honest case, we starts with the player-$\mathbf{x}$. Following the analysis of the convex functions in Appendix C.3, we decompose its regret by

$$\textsc{Reg}_T^{\mathbf{x}} \leq \underbrace{\sum_{t=1}^T \langle \mathbf{g}_t^{\mathbf{x}}, \mathbf{x}_t - \mathbf{x}_{t,i^\star}\rangle}_{\textsc{Meta-Reg}^{\mathbf{x}}} + \underbrace{\sum_{t=1}^T \langle \mathbf{g}_t^{\mathbf{x}}, \mathbf{x}_{t,i^\star} - \mathbf{x}^\star\rangle}_{\textsc{Base-Reg}^{\mathbf{x}}}. \tag{D.26}$$

For the meta regret, combining Eq. (A.3) and Eq. (A.7), we have

$$\begin{aligned}
\textsc{Meta-Reg}^{\mathbf{x}} \leq{} & \frac{Z^{\mathbf{x}}}{\varepsilon_{j^\star}^{\text{TOP}}} \log \frac{N}{3C_0^2(\varepsilon_{j^\star}^{\text{TOP}})^2} + 32 Z^{\mathbf{x}} \varepsilon_{j^\star}^{\text{TOP}} V_\star + \gamma_{\mathbf{x}}^{\text{MID}} S_{T,i^\star}^{\mathbf{x}} - \frac{C_0}{2} S_T^{\text{TOP},\mathbf{x}} \\
& - \gamma_{\mathbf{x}}^{\text{MID}} \sum_{t=2}^{T} \sum_{j=1}^{M} q_{t,j}^{\text{TOP},\mathbf{x}} \sum_{i=1}^{N} q_{t,j,i}^{\text{MID},\mathbf{x}} \left\|\mathbf{x}_{t,i} - \mathbf{x}_{t-1,i}\right\|^2 \\
& - \gamma_{\mathbf{y}}^{\text{TOP}} \sum_{t=2}^{T} \sum_{j=1}^{M} q_{t,j}^{\text{TOP},\mathbf{x}} \|\boldsymbol{q}_{t,j}^{\text{MID},\mathbf{x}} - \boldsymbol{q}_{t-1,j}^{\text{MID},\mathbf{x}}\|_1^2 \\
\leq{} & \frac{Z^{\mathbf{x}}}{\varepsilon_{j^\star}^{\text{TOP}}} \log \frac{N}{3C_0^2(\varepsilon_{j^\star}^{\text{TOP}})^2} + \frac{64\varepsilon_{j^\star}^{\text{TOP}}}{Z^{\mathbf{x}}} \sum_{t=1}^{T} \left\|\mathbf{g}_t^{\mathbf{x}} - \mathbf{g}_{t-1}^{\mathbf{x}}\right\|^2 + \left(\frac{64G^2}{Z^{\mathbf{x}}} + \gamma_{\mathbf{x}}^{\text{MID}}\right) S_{T,i^\star}^{\mathbf{x}} - \frac{C_0}{2} S_T^{\text{TOP},\mathbf{x}} \\
& + \frac{64G^2}{Z^{\mathbf{x}}} S_T^{\mathbf{x}} - \gamma_{\mathbf{x}}^{\text{MID}} \sum_{t=2}^{T} \sum_{j=1}^{M} q_{t,j}^{\text{TOP},\mathbf{x}} \sum_{i=1}^{N} q_{t,j,i}^{\text{MID},\mathbf{x}} \left\|\mathbf{x}_{t,i} - \mathbf{x}_{t-1,i}\right\|^2 \\
& - \gamma_{\mathbf{y}}^{\text{TOP}} \sum_{t=2}^{T} \sum_{j=1}^{M} q_{t,j}^{\text{TOP},\mathbf{x}} \|\boldsymbol{q}_{t,j}^{\text{MID},\mathbf{x}} - \boldsymbol{q}_{t-1,j}^{\text{MID},\mathbf{x}}\|_1^2 \\
\leq{} & \frac{Z^{\mathbf{x}}}{\varepsilon_{j^\star}^{\text{TOP}}} \log \frac{N}{3C_0^2(\varepsilon_{j^\star}^{\text{TOP}})^2} + \frac{32}{Z^{\mathbf{x}}} S_T^{\mathbf{x}} + \frac{32}{Z^{\mathbf{x}}} S_T^{\mathbf{y}} + \left(\frac{64G^2}{Z^{\mathbf{x}}} + \gamma_{\mathbf{x}}^{\text{MID}}\right) S_{T,i^\star}^{\mathbf{x}} - \frac{C_0}{2} S_T^{\text{TOP},\mathbf{x}} \\
& + \frac{64G^2}{Z^{\mathbf{x}}} S_T^{\mathbf{x}} - \gamma_{\mathbf{x}}^{\text{MID}} \sum_{t=2}^{T} \sum_{j=1}^{M} q_{t,j}^{\text{TOP},\mathbf{x}} \sum_{i=1}^{N} q_{t,j,i}^{\text{MID},\mathbf{x}} \left\|\mathbf{x}_{t,i} - \mathbf{x}_{t-1,i}\right\|^2 \\
& - \gamma_{\mathbf{x}}^{\text{TOP}} \sum_{t=2}^{T} \sum_{j=1}^{M} q_{t,j}^{\text{TOP},\mathbf{x}} \|\boldsymbol{q}_{t,j}^{\text{MID},\mathbf{x}} - \boldsymbol{q}_{t-1,j}^{\text{MID},\mathbf{x}}\|_1^2,
\end{aligned} \tag{D.27}$$

where the third step is by $\left\|\mathbf{g}_t^{\mathbf{x}} - \mathbf{g}_{t-1}^{\mathbf{x}}\right\|^2 \leq 2\left\|\mathbf{x}_t - \mathbf{x}_{t-1}\right\|^2 + 2\left\|\mathbf{y}_t - \mathbf{y}_{t-1}\right\|^2$ and $\varepsilon_{j^\star}^{\text{TOP}} \leq 1/2$.

For the base regret, by Lemma 21, we have

$$\begin{aligned}
\textsc{Base-Reg}^{\mathbf{x}} \leq{} & 5\sqrt{2}\sqrt{1 + \sum_{t=1}^{T} \left\|\mathbf{g}_t^{\mathbf{x}} - \mathbf{g}_{t-1}^{\mathbf{x}}\right\|^2} + \gamma_{\mathbf{x}}^{\text{TOP}} - \frac{1}{4}\gamma_{\mathbf{x}}^{\text{TOP}} S_{T,i^\star}^{\mathbf{x}} + \mathcal{O}(1) \\
\leq{} & 5\sqrt{2} \sum_{t=1}^{T} \left\|\mathbf{g}_t^{\mathbf{x}} - \mathbf{g}_{t-1}^{\mathbf{x}}\right\|^2 + \gamma_{\mathbf{x}}^{\text{TOP}} - \frac{1}{4}\gamma_{\mathbf{x}}^{\text{TOP}} S_{T,i^\star}^{\mathbf{x}} + \mathcal{O}(1) \\
\leq{} & 10\sqrt{2} S_T^{\mathbf{x}} + 10\sqrt{2} S_T^{\mathbf{y}} \gamma_{\mathbf{x}}^{\text{TOP}} - \frac{1}{4}\gamma_{\mathbf{x}}^{\text{TOP}} S_{T,i^\star}^{\mathbf{x}} + \mathcal{O}(1),
\end{aligned} \tag{D.28}$$

where the second step is by AM-GM inequality (Lemma 18) and the third step is by $\left\|\mathbf{g}_t^{\mathbf{x}} - \mathbf{g}_{t-1}^{\mathbf{x}}\right\|^2 \leq 2\left\|\mathbf{x}_t - \mathbf{x}_{t-1}\right\|^2 + 2\left\|\mathbf{y}_t - \mathbf{y}_{t-1}\right\|^2$. Plugging Eq. (D.27) and Eq. (D.28) into Eq. (D.26), we obtain

$$\begin{aligned}
\textsc{Reg}_T^{\mathbf{x}} \leq{} & \left(\frac{32}{Z^{\mathbf{x}}} + 10\sqrt{2}\right) S_T^{\mathbf{y}} + \left(\frac{64G^2}{Z^{\mathbf{x}}} + \gamma_{\mathbf{x}}^{\text{MID}} - \frac{1}{4}\kappa\right) S_{T,i^\star}^{\mathbf{x}} + 2\kappa - \frac{C_0}{2} S_T^{\text{TOP},\mathbf{x}} \\
& + \left(\frac{32 + 64G^2}{Z^{\mathbf{x}}} + 10\sqrt{2}\right) S_T^{\mathbf{x}} - \gamma_{\mathbf{x}}^{\text{MID}} \sum_{t=2}^{T} \sum_{j=1}^{M} q_{t,j}^{\text{TOP},\mathbf{x}} \sum_{i=1}^{N} q_{t,j,i}^{\text{MID},\mathbf{x}} \left\|\mathbf{x}_{t,i} - \mathbf{x}_{t-1,i}\right\|^2
\end{aligned}$$

$$- \gamma_{\mathbf{x}}^{\text{TOP}} \sum_{t=2}^{T}\sum_{j=1}^{M} q_{t,j}^{\text{TOP},\mathbf{x}} \|\boldsymbol{q}_{t,j}^{\text{MID},\mathbf{x}} - \boldsymbol{q}_{t-1,j}^{\text{MID},\mathbf{x}}\|_1^2 + \mathcal{O}(1)$$

$$\leq \left(\frac{32}{Z^{\mathbf{x}}} + 10\sqrt{2}\right) S_T^{\mathbf{y}} + 2\kappa - \frac{C_0}{2} S_T^{\text{TOP},\mathbf{x}} + \left(\frac{32+64G^2}{Z^{\mathbf{x}}} + 10\sqrt{2}\right) S_T^{\mathbf{x}} + \mathcal{O}(1)$$

$$- \gamma_{\mathbf{x}}^{\text{MID}} \sum_{t=2}^{T}\sum_{j=1}^{M} q_{t,j}^{\text{TOP},\mathbf{x}} \sum_{i=1}^{N} q_{t,j,i}^{\text{MID},\mathbf{x}} \|\mathbf{x}_{t,i} - \mathbf{x}_{t-1,i}\|^2 - \gamma_{\mathbf{x}}^{\text{TOP}} \sum_{t=2}^{T}\sum_{j=1}^{M} q_{t,j}^{\text{TOP},\mathbf{x}} \|\boldsymbol{q}_{t,j}^{\text{MID},\mathbf{x}} - \boldsymbol{q}_{t-1,j}^{\text{MID},\mathbf{x}}\|_1^2, \tag{D.29}$$

where the last step is by choosing $\kappa \geq 256G^2 + 4\gamma_{\mathbf{x}}^{\text{MID}}$. Symmetrically, for player-$\mathbf{y}$, we can obtain a similar bound

$$\text{Reg}_T^{\mathbf{y}} \leq \left(\frac{32}{Z^{\mathbf{y}}} + 10\sqrt{2}\right) S_T^{\mathbf{x}} + \left(\frac{64G^2}{Z^{\mathbf{y}}} + \gamma_{\mathbf{y}}^{\text{MID}} - \frac{1}{4}\kappa\right) S_{T,i^\star}^{\mathbf{y}} + 2\kappa - \frac{C_0}{2} S_T^{\text{TOP},\mathbf{y}}$$

$$+ \left(\frac{32+64G^2}{Z^{\mathbf{y}}} + 10\sqrt{2}\right) S_T^{\mathbf{y}} - \gamma_{\mathbf{y}}^{\text{MID}} \sum_{t=2}^{T}\sum_{j=1}^{M} q_{t,j}^{\text{TOP},\mathbf{y}} \sum_{i=1}^{N} q_{t,j,i}^{\text{MID},\mathbf{y}} \|\mathbf{y}_{t,i} - \mathbf{y}_{t-1,i}\|^2$$

$$- \gamma_{\mathbf{x}}^{\text{TOP}} \sum_{t=2}^{T}\sum_{j=1}^{M} q_{t,j}^{\text{TOP},\mathbf{y}} \|\boldsymbol{q}_{t,j}^{\text{MID},\mathbf{y}} - \boldsymbol{q}_{t-1,j}^{\text{MID},\mathbf{y}}\|_1^2 + \mathcal{O}(1)$$

$$\leq \left(\frac{32}{Z^{\mathbf{y}}} + 10\sqrt{2}\right) S_T^{\mathbf{x}} + 2\kappa - \frac{C_0}{2} S_T^{\text{TOP},\mathbf{y}} + \left(\frac{32+64G^2}{Z^{\mathbf{y}}} + 10\sqrt{2}\right) S_T^{\mathbf{y}} + \mathcal{O}(1)$$

$$- \gamma_{\mathbf{y}}^{\text{MID}} \sum_{t=2}^{T}\sum_{j=1}^{M} q_{t,j}^{\text{TOP},\mathbf{y}} \sum_{i=1}^{N} q_{t,j,i}^{\text{MID},\mathbf{y}} \|\mathbf{y}_{t,i} - \mathbf{y}_{t-1,i}\|^2 - \gamma_{\mathbf{x}}^{\text{TOP}} \sum_{t=2}^{T}\sum_{j=1}^{M} q_{t,j}^{\text{TOP},\mathbf{y}} \|\boldsymbol{q}_{t,j}^{\text{MID},\mathbf{y}} - \boldsymbol{q}_{t-1,j}^{\text{MID},\mathbf{y}}\|_1^2, \tag{D.30}$$

where the last step is by choosing $\kappa \geq 256 + 4\gamma_{\mathbf{y}}^{\text{MID}}$. Combining Eq. (D.29) and Eq. (D.30) and letting $C_{31} = \frac{32+64G^2}{Z^{\mathbf{x}}} + \frac{32}{Z^{\mathbf{y}}} + 20\sqrt{2}$ and $C_{32} = \frac{32+64G^2}{Z^{\mathbf{y}}} + \frac{32}{Z^{\mathbf{x}}} + 20\sqrt{2}$, we obtain

$$\text{Reg}_T^{\mathbf{x}} + \text{Reg}_T^{\mathbf{y}} \leq C_{31} S_T^{\mathbf{x}} + C_{32} S_T^{\mathbf{y}} - \frac{C_0}{2}(S_T^{\text{TOP},\mathbf{x}} + S_T^{\text{TOP},\mathbf{y}}) + \mathcal{O}(1)$$

$$- \gamma_{\mathbf{x}}^{\text{MID}} \sum_{t=2}^{T}\sum_{j=1}^{M} q_{t,j}^{\text{TOP},\mathbf{x}} \sum_{i=1}^{N} q_{t,j,i}^{\text{MID},\mathbf{x}} \|\mathbf{x}_{t,i} - \mathbf{x}_{t-1,i}\|^2 - \gamma_{\mathbf{x}}^{\text{TOP}} \sum_{t=2}^{T}\sum_{j=1}^{M} q_{t,j}^{\text{TOP},\mathbf{x}} \|\boldsymbol{q}_{t,j}^{\text{MID},\mathbf{x}} - \boldsymbol{q}_{t-1,j}^{\text{MID},\mathbf{x}}\|_1^2$$

$$- \gamma_{\mathbf{y}}^{\text{MID}} \sum_{t=2}^{T}\sum_{j=1}^{M} q_{t,j}^{\text{TOP},\mathbf{y}} \sum_{i=1}^{N} q_{t,j,i}^{\text{MID},\mathbf{y}} \|\mathbf{y}_{t,i} - \mathbf{y}_{t-1,i}\|^2 - \gamma_{\mathbf{y}}^{\text{TOP}} \sum_{t=2}^{T}\sum_{j=1}^{M} q_{t,j}^{\text{TOP},\mathbf{y}} \|\boldsymbol{q}_{t,j}^{\text{MID},\mathbf{y}} - \boldsymbol{q}_{t-1,j}^{\text{MID},\mathbf{y}}\|_1^2$$

$$\leq \left(8C_{31} - \frac{C_0}{2}\right) S_T^{\text{TOP},\mathbf{x}} + \left(8C_{32} - \frac{C_0}{2}\right) S_T^{\text{TOP},\mathbf{y}} + \mathcal{O}(1)$$

$$+ (2C_{31} - \gamma_{\mathbf{x}}^{\text{MID}}) \sum_{t=2}^{T}\sum_{j=1}^{M} q_{t,j}^{\text{TOP},\mathbf{x}} \sum_{i=1}^{N} q_{t,j,i}^{\text{MID},\mathbf{x}} \|\mathbf{x}_{t,i} - \mathbf{x}_{t-1,i}\|^2$$

$$+ (2C_{32} - \gamma_{\mathbf{y}}^{\text{MID}}) \sum_{t=2}^{T}\sum_{j=1}^{M} q_{t,j}^{\text{TOP},\mathbf{y}} \sum_{i=1}^{N} q_{t,j,i}^{\text{MID},\mathbf{y}} \|\mathbf{y}_{t,i} - \mathbf{y}_{t-1,i}\|^2$$

$$+ (8C_{31} - \gamma_{\mathbf{x}}^{\text{TOP}}) \sum_{t=2}^{T} \sum_{j=1}^{M} q_{t,j}^{\text{TOP},\mathbf{x}} \|\boldsymbol{q}_{t,j}^{\text{MID},\mathbf{x}} - \boldsymbol{q}_{t-1,j}^{\text{MID},\mathbf{x}}\|_1^2$$
$$+ (8C_{32} - \gamma_{\mathbf{y}}^{\text{TOP}}) \sum_{t=2}^{T} \sum_{j=1}^{M} q_{t,j}^{\text{TOP},\mathbf{y}} \|\boldsymbol{q}_{t,j}^{\text{MID},\mathbf{y}} - \boldsymbol{q}_{t-1,j}^{\text{MID},\mathbf{y}}\|_1^2 \le \mathcal{O}(1),$$

where the first step is due to Lemma 5 and the second step is by choosing $\gamma_{\mathbf{x}}^{\text{MID}} \ge 2C_{31}$, $\gamma_{\mathbf{y}}^{\text{MID}} \ge 2C_{32}$, $\gamma_{\mathbf{x}}^{\text{TOP}} \ge 8C_{31}$, $\gamma_{\mathbf{y}}^{\text{TOP}} \ge 8C_{32}$, and $C_0 \ge 16\max\{C_{31}, C_{32}\}$.

At last, we determine the specific values of C_0, $\gamma_{\mathbf{x}}^{\text{TOP}}$, $\gamma_{\mathbf{x}}^{\text{MID}}$, $\gamma_{\mathbf{y}}^{\text{TOP}}$, $\gamma_{\mathbf{y}}^{\text{MID}}$. These parameters need to satisfy the following requirements:

$$C_0 \ge 1,\ C_0 \ge 8D,\ C_0 \ge 4\gamma_{\mathbf{x}}^{\text{TOP}},\ C_0 \ge 4\gamma_{\mathbf{y}}^{\text{TOP}},\ C_0 \ge 16C_{31},\ C_0 \ge 16C_{32},\ \gamma_{\mathbf{x}}^{\text{MID}} \ge 2C_{31},$$
$$\gamma_{\mathbf{y}}^{\text{MID}} \ge 2C_{32},\ \gamma_{\mathbf{x}}^{\text{TOP}} \ge 8C_{31},\ \gamma_{\mathbf{y}}^{\text{TOP}} \ge 8C_{32}.$$

As a result, we set

$$C_0 = \max\left\{1, 8D, 4\gamma^{\text{TOP}}, 16C_{31}, 16C_{32}, 4\gamma_{\mathbf{x}}^{\text{TOP}}, 4\gamma_{\mathbf{y}}^{\text{TOP}}\right\},$$
$$\gamma_{\mathbf{x}}^{\text{MID}} = \gamma_{\mathbf{y}}^{\text{MID}} = 128 + 128G^2 + 40\sqrt{2}, \quad \gamma_{\mathbf{x}}^{\text{TOP}} = \gamma_{\mathbf{y}}^{\text{TOP}} = 512 + 512G^2 + 160\sqrt{2},$$

where $C_{31} = \frac{32+64G^2}{Z^{\mathbf{x}}} + \frac{32}{Z^{\mathbf{y}}} + 20\sqrt{2}$, $C_{32} = \frac{32+64G^2}{Z^{\mathbf{y}}} + \frac{32}{Z^{\mathbf{x}}} + 20\sqrt{2}$, $Z^{\mathbf{x}} = \max\{GD + \gamma_{\mathbf{x}}^{\text{MID}} D^2, 1 + \gamma_{\mathbf{x}}^{\text{MID}} D^2 + 2\gamma_{\mathbf{x}}^{\text{TOP}}\}$, and $Z^{\mathbf{y}} = \max\{GD + \gamma_{\mathbf{y}}^{\text{MID}} D^2, 1 + \gamma_{\mathbf{y}}^{\text{MID}} D^2 + 2\gamma_{\mathbf{y}}^{\text{TOP}}\}$. The proof is finished. ■

D.6 Proof of Theorem 8

Proof In this proof, note that we do not need to analyze the convex case because the convex base learner is naturally anytime by using a self-confident tuning step size, as in Lemma 21. Therefore, in the rest of the proof, we focus on the cases of λ-strongly convex and α-exp-concave functions.

Regret Decomposition. For λ-*strongly convex* functions, recall that $i^\star$ denotes the index of the best base learner whose strong convexity coefficient satisfies $\lambda_{i^\star} \le \lambda \le 2\lambda_{i^\star}$. We consider two cases at the τ-round: *(i)* the $i^\star$-th base learner is not activated (i.e., $\tau < s_{i^\star}^{\text{sc}}$); and *(ii)* the $i^\star$-th base learner is activated (i.e., $\tau \ge s_{i^\star}^{\text{sc}}$). For case *(i)*, we obtain

$$\text{Reg}_\tau = \sum_{t=1}^{\tau} f_t(\mathbf{x}_t) - \sum_{t=1}^{\tau} f_t(\mathbf{x}^\star) \le \tau GD \le (s_{i^\star}^{\text{sc}} - 1)GD \le \mathcal{O}\left(\frac{1}{\lambda_{i^\star}}\right) \le \mathcal{O}\left(\frac{1}{\lambda}\right),$$

where the third step follows from the fact that $\tau \le s_{i^\star}^{\text{sc}} - 1$, since the corresponding base learner has not been activated at the τ-th round. The fourth step is due to the activation condition of $s_{i^\star}^{\text{sc}} - 1 < \frac{1}{\lambda_{i^\star}}$. For case *(ii)*, we first decompose the regret into two parts, where the first part corresponds to case *(i)*, and the second one refers to regret after the $i^\star$-th base learner is activated:

$$\text{Reg}_\tau = \underbrace{\sum_{t=1}^{s_{i^\star}^{\text{sc}}-1} f_t(\mathbf{x}_t) - \sum_{t=1}^{s_{i^\star}^{\text{sc}}-1} f_t(\mathbf{x}^\star)}_{\text{term (a)}} + \underbrace{\sum_{t=s_{i^\star}^{\text{sc}}}^{\tau} f_t(\mathbf{x}_t) - \sum_{t=s_{i^\star}^{\text{sc}}}^{\tau} f_t(\mathbf{x}^\star)}_{\text{term (b)}}. \tag{D.31}$$

TERM (A) can be bounded by $\mathcal{O}(1/\lambda)$ as in case *(i)*. We then decompose TERM (B) into the following two parts:

$$\text{TERM (B)} \leq \underbrace{\sum_{t=s_{i^\star}^{\text{sc}}}^{\tau} \langle \mathbf{g}_t, \mathbf{x}_t - \mathbf{x}_{t,i^\star}\rangle - \frac{\lambda_{i^\star}}{4}\sum_{t=s_{i^\star}^{\text{sc}}}^{\tau} \|\mathbf{x}_t - \mathbf{x}_{t,i^\star}\|^2}_{\text{META-REG}} \qquad (\text{by } \lambda_{i^\star} \leq \lambda \leq 2\lambda_{i^\star})$$

$$+ \underbrace{\sum_{t=s_{i^\star}^{\text{sc}}}^{\tau} h_{t,i^\star}^{\text{sc}}(\mathbf{x}_{t,i^\star}) - \sum_{t=s_{i^\star}^{\text{sc}}}^{\tau} h_{t,i^\star}^{\text{sc}}(\mathbf{x}^\star)}_{\text{BASE-REG}} - \frac{1}{2}\sum_{t=s_{i^\star}^{\text{sc}}}^{\tau} \mathcal{D}_{f_t}(\mathbf{x}^\star, \mathbf{x}_t). \qquad \text{(D.32)}$$

For α-*exp-concave* functions, recall that $i^\star$ denotes the index of the best base learner whose exp-concave coefficient satisfies $\alpha_{i^\star} \leq \alpha \leq 2\alpha_{i^\star}$. Similar to the strongly convex functions, we consider two cases at the τ-round: *(i)* the $i^\star$-th base learner is not activated (i.e., $\tau < s_{i^\star}^{\text{exp}}$); and *(ii)* the $i^\star$-th base learner is activated (i.e., $\tau \geq s_{i^\star}^{\text{exp}}$). For case *(i)*, we obtain

$$\text{REG}_\tau = \sum_{t=1}^{\tau} f_t(\mathbf{x}_t) - \sum_{t=1}^{\tau} f_t(\mathbf{x}^\star) \leq \tau GD \leq (s_{i^\star}^{\text{exp}} - 1)GD \leq \mathcal{O}\left(\frac{1}{\alpha_{i^\star}}\right) \leq \mathcal{O}\left(\frac{d}{\alpha}\right),$$

where the third step follows from the fact that $\tau \leq s_{i^\star}^{\text{exp}} - 1$, since the corresponding base learner has not been activated at this time. The fourth step is due to the activation condition of $s_{i^\star}^{\text{exp}} - 1 < \frac{1}{\alpha_{i^\star}}$. For case *(ii)*, we first decompose the regret into two parts, where the first part corresponds to case *(i)*, and the second one refers to regret after the $i^\star$-th base learner is activated:

$$\text{REG}_\tau = \underbrace{\sum_{t=1}^{s_{i^\star}^{\text{exp}}-1} f_t(\mathbf{x}_t) - \sum_{t=1}^{s_{i^\star}^{\text{exp}}-1} f_t(\mathbf{x}^\star)}_{\text{TERM (A)}} + \underbrace{\sum_{t=s_{i^\star}^{\text{exp}}}^{\tau} f_t(\mathbf{x}_t) - \sum_{t=s_{i^\star}^{\text{exp}}}^{\tau} f_t(\mathbf{x}^\star)}_{\text{TERM (B)}}. \qquad \text{(D.33)}$$

We bound TERM (A) by $\mathcal{O}\left(\frac{d}{\alpha}\right)$ as in case *(i)* and decompose TERM (B) into the following two parts:

$$\text{TERM (B)} \leq \underbrace{\sum_{t=s_{i^\star}^{\text{exp}}}^{\tau} \langle \mathbf{g}_t, \mathbf{x}_t - \mathbf{x}_{t,i^\star}\rangle - \frac{\alpha_{i^\star}}{4}\sum_{t=s_{i^\star}^{\text{exp}}}^{\tau} \langle \mathbf{g}_t, \mathbf{x}_t - \mathbf{x}_{t,i^\star}\rangle^2}_{\text{META-REG}} \qquad (\text{by } \alpha_{i^\star} \leq \alpha \leq 2\alpha_{i^\star})$$

$$+ \underbrace{\sum_{t=s_{i^\star}^{\text{exp}}}^{\tau} h_{t,i^\star}^{\text{exp}}(\mathbf{x}_{t,i^\star}) - \sum_{t=s_{i^\star}^{\text{exp}}}^{\tau} h_{t,i^\star}^{\text{exp}}(\mathbf{x}^\star)}_{\text{BASE-REG}} - \frac{1}{2}\sum_{t=s_{i^\star}^{\text{exp}}}^{\tau} \mathcal{D}_{f_t}(\mathbf{x}^\star, \mathbf{x}_t). \qquad \text{(D.34)}$$

Meta Regret Analysis. We adopt Optimistic-Adapt-ML-Prod variant as the meta learner, and present its regret analysis below for self-containedness.

Lemma 9 (Simplfied Theorem 7 of Xie et al. (2024)). *Denoting by $\mathcal{A}_t$ the active expert set at time t, $\boldsymbol{p}_t \in \Delta_{|\mathcal{A}_t|}$ the algorithm's weights, $\boldsymbol{\ell}_t \in \mathbb{R}^{|\mathcal{A}_t|}$ the loss vector, and $\boldsymbol{m}_t \in \mathbb{R}^{|\mathcal{A}_t|}$*

the optimism. Assuming that the i-th expert participates in prediction during time $[a,b]$ and choosing the learning rate optimally as Eq. (6.12), the regret of Optimistic-Adapt-ML-Prod variant with respect to expert i satisfies

$$\sum_{t=a}^{b}\langle \boldsymbol{\ell}_t, \boldsymbol{p}_t - \boldsymbol{e}_i\rangle \le C_{33}\sqrt{1+\sum_{t=a}^{b}(r_{t,i}-m_{t,i})^2},$$

where $\boldsymbol{e}_i$ denotes the i-th standard basis vector, $C_{33} = \mathcal{O}(\log N_b + \log(1+\log b))$, and N_b represents the total number of base learners initialized till time $b+1$.

For λ-*strongly convex* functions, the meta regret in Eq. (D.32) can be bounded as

$$\begin{aligned}
\textsc{Meta-Reg} &\le C_{33}\sqrt{1+\sum_{t=s_{i^\star}^{\mathrm{sc}}}^{\tau}\langle \mathbf{g}_t, \mathbf{x}_t - \mathbf{x}_{t,i^\star}\rangle^2} - \frac{\lambda_{i^\star}}{4}\sum_{t=s_{i^\star}^{\mathrm{sc}}}^{\tau}\|\mathbf{x}_t - \mathbf{x}_{t,i^\star}\|^2\\
&\le C_{33}\sqrt{1+G^2\sum_{t=s_{i^\star}^{\mathrm{sc}}}^{\tau}\|\mathbf{x}_t - \mathbf{x}_{t,i^\star}\|^2} - \frac{\lambda_{i^\star}}{4}\sum_{t=s_{i^\star}^{\mathrm{sc}}}^{\tau}\|\mathbf{x}_t - \mathbf{x}_{t,i^\star}\|^2\\
&\le \mathcal{O}(C_{34}) + \left(\frac{C_{33}G^2}{2C_{34}} - \frac{\lambda_{i^\star}}{4}\right)\sum_{t=s_{i^\star}^{\mathrm{sc}}}^{\tau}\|\mathbf{x}_t - \mathbf{x}_{t,i^\star}\|^2, \qquad \text{(D.35)}
\end{aligned}$$

where the first step follows from Lemma 9 by choosing $a = s_{i^\star}^{\mathrm{sc}}$ and $b=\tau$, the last step uses AM-GM inequality (Lemma 18) and omits the ignorable additive C_{33} terms. $N_\tau = \mathcal{O}(\log \tau)$ and we omit the $\mathcal{O}(\log\log\tau)$ term. C_{34} is a constant to be specified.

For α-*exp-concave* functions, the meta regret in Eq. (D.34) can be bounded as

$$\begin{aligned}
\textsc{Meta-Reg} &\le C_{33}\sqrt{1+\sum_{t=s_{i^\star}^{\exp}}^{\tau}\langle \mathbf{g}_t, \mathbf{x}_t - \mathbf{x}_{t,i^\star}\rangle^2} - \frac{\alpha_{i^\star}}{4}\sum_{t=s_{i^\star}^{\exp}}^{\tau}\langle \mathbf{g}_t, \mathbf{x}_t - \mathbf{x}_{t,i^\star}\rangle^2\\
&\le C_{33}\sqrt{1+\sum_{t=s_{i^\star}^{\exp}}^{\tau}\langle \mathbf{g}_t, \mathbf{x}_t - \mathbf{x}_{t,i^\star}\rangle^2} - \frac{\alpha_{i^\star}}{4}\sum_{t=s_{i^\star}^{\exp}}^{\tau}\langle \mathbf{g}_t, \mathbf{x}_t - \mathbf{x}_{t,i^\star}\rangle^2\\
&\le \mathcal{O}(C_{36}) + \left(\frac{C_{33}}{2C_{36}} - \frac{\alpha_{i^\star}}{4}\right)\sum_{t=s_{i^\star}^{\exp}}^{\tau}\langle \mathbf{g}_t, \mathbf{x}_t - \mathbf{x}_{t,i^\star}\rangle^2, \qquad \text{(D.36)}
\end{aligned}$$

where the first step follows from Lemma 9 by choosing $a = s_{i^\star}^{\exp}$ and $b=\tau$, the last step uses AM-GM inequality (Lemma 18) and omits the ignorable additive C_{33} terms. It's because $N_\tau = \mathcal{O}(\log\tau)$ and we omit $\log\log\tau$ terms. C_{36} is a constant to be specified.

Base Regret Analysis. Following the analysis structure of Appendix C.3, we first provide different decompositions of the empirical gradient variation defined on surrogates for strongly convex and exp-concave functions, respectively, and then analyze the base regret in the corresponding cases. For λ-*strongly convex* functions, since the $i^\star$-th base learner is activated at time $s_{i^\star}^{\mathrm{sc}}$, we can directly bound the empirical gradient variation on surrogates, i.e., $\bar{V}^{\text{sc}}_{[s^{\text{sc}}_{i^\star},\tau],i^\star} \triangleq \sum_{t=s^{\text{sc}}_{i^\star}+1}^{\tau} \|\nabla h^{\text{sc}}_{t,i^\star}(\mathbf{x}_{t,i^\star}) - \nabla h^{\text{sc}}_{t-1,i^\star}(\mathbf{x}_{t-1,i^\star})\|^2$ by (C.21)

$$\bar{V}^{\text{sc}}_{[s^{\text{sc}}_{i^\star},\tau],i^\star} \le 9V_{[s^{\text{sc}}_{i^\star},\tau]} + 36L \sum_{t=s^{\text{sc}}_{i^\star}}^{\tau} \mathcal{D}_{f_t}(\mathbf{x}^\star, \mathbf{x}_t) + 2\lambda^2_{i^\star} \sum_{t=s^{\text{sc}}_{i^\star}}^{\tau} \|\mathbf{x}_{t,i^\star} - \mathbf{x}_t\|^2,$$

where $V_{[s^{\text{sc}}_{i^\star},\tau]} = \sum_{t=s^{\text{sc}}_{i^\star}+1}^{\tau} \sup_{\mathbf{x}\in\mathcal{X}} \|\nabla f_t(\mathbf{x}) - f_{t-1}(\mathbf{x})\|^2$.

For α-*exp-concave* functions, by (C.21), we can similarly bound the empirical gradient variation on surrogates, i.e., $\bar{V}^{\text{exp}}_{[s^{\text{exp}}_{i^\star},\tau],i^\star} \triangleq \sum_{t=s^{\text{exp}}_{i^\star}+1}^{\tau} \|\nabla h^{\text{exp}}_{t,i^\star}(\mathbf{x}_{t,i^\star}) - \nabla h^{\text{exp}}_{t-1,i^\star}(\mathbf{x}_{t-1,i^\star})\|^2$, by

$$\bar{V}^{\text{exp}}_{[s^{\text{exp}}_{i^\star},\tau],i^\star} \le 9V_{[s^{\text{exp}}_{i^\star},\tau]} + 36L \sum_{t=s^{\text{exp}}_{i^\star}}^{\tau} \mathcal{D}_{f_t}(\mathbf{x}^\star, \mathbf{x}_t) + 2\alpha^2_{i^\star} G^2 \sum_{t=s^{\text{exp}}_{i^\star}}^{\tau} \langle \mathbf{g}_t, \mathbf{x}_t - \mathbf{x}_{t,i^\star}\rangle^2,$$

where $V_{[s^{\text{exp}}_{i^\star},\tau]} = \sum_{t=s^{\text{exp}}_{i^\star}+1}^{\tau} \sup_{\mathbf{x}\in\mathcal{X}} \|\nabla f_t(\mathbf{x}) - f_{t-1}(\mathbf{x})\|^2$. To conclude, for different curvature types, we provide correspondingly different analysis of the empirical gradient variation on surrogates:

$$\bar{V}^{\{\text{sc},\text{exp}\}}_{[s^{\text{sc}}_{i^\star},\tau],i^\star} \le \begin{cases} 9V_{[s^{\text{sc}}_{i^\star},\tau]} + 36L \sum_{t=s^{\text{sc}}_{i^\star}}^{\tau} \mathcal{D}_{f_t}(\mathbf{x}^\star, \mathbf{x}_t) + 2\lambda^2_{i^\star} \sum_{t=s^{\text{sc}}_{i^\star}}^{\tau} \|\mathbf{x}_{t,i^\star} - \mathbf{x}_t\|^2, & (\lambda\text{-strongly convex}) \\ 9V_{[s^{\text{exp}}_{i^\star},\tau]} + 36L \sum_{t=s^{\text{exp}}_{i^\star}}^{\tau} \mathcal{D}_{f_t}(\mathbf{x}^\star, \mathbf{x}_t) + 2\alpha^2_{i^\star} G^2 \sum_{t=s^{\text{exp}}_{i^\star}}^{\tau} \langle \mathbf{g}_t, \mathbf{x}_t - \mathbf{x}_{t,i^\star}\rangle^2. & (\alpha\text{-exp-concave}) \end{cases} \tag{D.37}$$

In the following, we analyze the base regret for different curvature types. Since the empirical gradient variation shares a similar structure to that in Theorem 4, we can directly apply the corresponding result. For λ-*strongly convex* functions, according to Eq. (C.22), the base regret can be bounded as

$$\begin{aligned} \textsc{Base-Reg} \le\ & \mathcal{O}\left(\frac{1}{\lambda} \log V_{[s^{\text{sc}}_{i^\star},\tau]}\right) + \frac{576 G^2 L}{C_{35}} \sum_{t=s^{\text{sc}}_{i^\star}}^{\tau} \mathcal{D}_{f_t}(\mathbf{x}^\star, \mathbf{x}_t) \\ & + \frac{32 G^2}{C_{35}} \sum_{t=s^{\text{sc}}_{i^\star}}^{\tau} \|\mathbf{x}_{t,i^\star} - \mathbf{x}_t\|^2 + \mathcal{O}(\log C_{35}). \end{aligned} \tag{D.38}$$

For α-*exp-concave* functions, according to Eq. (C.23), the base regret can be bounded as

$$\begin{aligned} \textsc{Base-Reg} \le\ & \mathcal{O}\left(\frac{d}{\alpha} \log V_{[s^{\text{exp}}_{i^\star},\tau]}\right) + \frac{72L}{C_{37}} \sum_{t=s^{\text{exp}}_{i^\star}}^{\tau} \mathcal{D}_{f_t}(\mathbf{x}^\star, \mathbf{x}_t) \\ & + \frac{4G^2}{C_{37}} \sum_{t=s^{\text{exp}}_{i^\star}}^{\tau} \langle \mathbf{g}_t, \mathbf{x}_t - \mathbf{x}_{t,i^\star}\rangle^2 + \mathcal{O}(\log C_{37}). \end{aligned} \tag{D.39}$$

Overall Regret Analysis. For λ-*strongly convex* functions, by combining Eq. (D.31), Eq. (D.32), Eq. (D.35), and Eq. (D.38), we obtain

$$\begin{aligned}\textsc{Reg}_\tau \le{}& \mathcal{O}\left(\frac{1}{\lambda}\right) + \mathcal{O}\left(\frac{1}{\lambda}\log V_{[s_{i^\star}^{\mathrm{sc}},\tau]}\right) + \left(\frac{576G^2L}{C_{35}} - \frac{1}{2}\right)\sum_{t=s_{i^\star}^{\mathrm{sc}}}^{\tau} \mathcal{D}_{f_t}(\mathbf{x}^\star, \mathbf{x}_t) \\ &+ \left(\frac{32G^2}{C_{35}} + \frac{C_{33}G^2}{2C_{34}} - \frac{\lambda_{i^\star}}{4}\right)\sum_{t=s_{i^\star}^{\mathrm{sc}}}^{\tau} \|\mathbf{x}_{t,i^\star} - \mathbf{x}_t\|^2 + \mathcal{O}\left(C_{34} + \log C_{35}\right) \le \mathcal{O}\left(\frac{1}{\lambda}\log V_\tau\right),\end{aligned}$$

where we choose $C_{34} = 4C_{33}G^2/\lambda_{i^\star}$ and $C_{35} = \max\{1, 256G^2/\lambda_{i^\star}, 1152G^2L\}$.

For α-*exp-concave* functions, combining Eq. (D.33), Eq. (D.34), Eq. (D.36), and Eq. (D.39),

$$\begin{aligned}\textsc{Reg}_\tau \le{}& \mathcal{O}\left(\frac{d}{\alpha}\right) + \mathcal{O}\left(\frac{d}{\alpha}\log V_{[s_{i^\star}^{\exp},\tau]}\right) + \left(\frac{72L}{C_{37}} - \frac{1}{2}\right)\sum_{t=s_{i^\star}^{\exp}}^{\tau} \mathcal{D}_{f_t}(\mathbf{x}^\star, \mathbf{x}_t) \\ &+ \left(\frac{4G^2}{C_{37}} + \frac{C_{33}}{2C_{36}} - \frac{\alpha_{i^\star}}{4}\right)\sum_{t=s_{i^\star}^{\exp}}^{\tau} \langle \mathbf{g}_t, \mathbf{x}_t - \mathbf{x}_{t,i^\star}\rangle^2 + \mathcal{O}\left(C_{36} + \log C_{37}\right) \le \mathcal{O}\left(\frac{d}{\alpha}\log V_\tau\right),\end{aligned}$$

where we choose $C_{36} = 4C_{33}/\alpha_{i^\star}$ and $C_{37} = \max\{1, 144L, 32G^2/\alpha_{i^\star}\}$. Note that the constants $C_{33}, C_{34}, C_{35}, C_{36}, C_{37}$ only exist in analysis and and hence our choices of them are feasible. ■

Appendix E. Technical Lemmas

In this section, we present several supporting lemmas used in proving our theoretical results. In Appendix E.1, we provide useful lemmas for the decomposition of two combined decisions and the parameter tuning. And in Appendix E.2, we analyze the stability-based negative terms of the base algorithms for different curvature types.

E.1 Useful Lemmas

In this part, we conclude some useful lemmas for bounding the gap between two combined decisions (Lemma 10 and Lemma 11), tuning the parameter (Lemma 12 and Lemma 13), and a useful summation (Lemma 14).

Lemma 10. *Under Assumption 1, if* $\mathbf{x} = \sum_{i=1}^N p_i \mathbf{x}_i, \mathbf{y} = \sum_{i=1}^N q_i \mathbf{y}_i$, *where* $\boldsymbol{p}, \boldsymbol{q} \in \Delta_N, \mathbf{x}_i, \mathbf{y}_i \in \mathcal{X}$ *for any* $i \in [N]$, *then it holds that*

$$\|\mathbf{x} - \mathbf{y}\|^2 \le 2\sum_{i=1}^N p_i \|\mathbf{x}_i - \mathbf{y}_i\|^2 + 2D^2\|\boldsymbol{p} - \boldsymbol{q}\|_1^2.$$

Lemma 11. *If* $\boldsymbol{w} = \sum_{i=1}^N q_i \boldsymbol{p}_i, \boldsymbol{w}' = \sum_{i=1}^N q_i' \boldsymbol{p}_i'$, *where* $\boldsymbol{q}, \boldsymbol{q}' \in \Delta_N$ *and* $\boldsymbol{p}_i, \boldsymbol{p}_i' \in \Delta_d$ *for any* $i \in [N]$, *then it holds that*

$$\|\boldsymbol{w} - \boldsymbol{w}'\|^2 \le 2\sum_{i=1}^N q_i \|\boldsymbol{p}_i - \boldsymbol{p}_i'\|_1^2 + 2\|\boldsymbol{q} - \boldsymbol{q}'\|_1^2.$$

Lemma 12. *For a step size pool of* $\mathcal{H}_\eta = \{\eta_k\}_{k\in[K]}$*, where* $\eta_1 = \frac{1}{2C_0} \geq \ldots \geq \eta_K = \frac{1}{2C_0 T}$*, if* $C_0 \geq \frac{\sqrt{X}}{2T}$*, there exists* $\eta \in \mathcal{H}_\eta$ *such that*

$$\frac{1}{\eta}\log\frac{Y}{\eta^2} + \eta X \leq 2C_0\log(4YC_0^2) + 4\sqrt{X\log(4XY)}.$$

Lemma 13. *Denoting by* $\eta_\star$ *the optimal step size, for a step size pool of* $\mathcal{H}_\eta = \{\eta_k\}_{k\in[K]}$*, where* $\eta_1 = \frac{1}{2C_0} \geq \ldots \geq \eta_K = \frac{1}{2C_0 T}$*, if* $C_0 \geq \frac{1}{2\eta_\star T}$*, there exists* $\eta \in \mathcal{H}_\eta$ *such that*

$$\frac{1}{\eta}\log\frac{Y}{\eta^2} \leq 2C_0\log(4YC_0^2) + \frac{2}{\eta_\star}\log\frac{4Y}{\eta_\star^2}.$$

Lemma 14. *For a sequence of* $\{a_t\}_{t=1}^T$ *and* b*, where* $a_t, b > 0$ *for any* $t \in [T]$*, denoting by* $a_{\max} \triangleq \max_t a_t$ *and* $A \triangleq \lceil b\sum_{t=1}^T a_t\rceil$*, we have*

$$\sum_{t=1}^T \frac{a_t}{bt} \leq \frac{a_{\max}}{b}(1 + \log A) + \frac{1}{b^2}.$$

Lemma 15. *For any* $a > 1, b > 0$*, it holds that* $\log(a+b) \leq \log(Ca) + \frac{b}{C}$ *for some* $C \geq 1$*.*

Lemma 16 (Corollary 5 of Orabona et al. (2012))**.** *If* $a, b, c, d, x > 0$ *satisfy* $x - d \leq a\log(bx + c)$*, then it holds that*

$$x - d \leq a\log\left(2ab\log\frac{2ab}{e} + 2bd + 2c\right).$$

Lemma 17 (Lemma 9 of Zhao et al. (2024))**.** *For any* $x, y, a, b > 0$ *satisfying* $x - y \leq \sqrt{ax} + b$*, it holds that*

$$x - y \leq \sqrt{ay + ab} + a + b.$$

Lemma 18 (AM-GM Inequality)**.** $\sqrt{xy} \leq \frac{ax}{2} + \frac{y}{2a}$ *for any* $x, y, a > 0$*.*

Proof [of Lemma 10] The term of $\|\mathbf{x} - \mathbf{y}\|^2$ can be decomposed as follows:

$$\begin{aligned}
\|\mathbf{x} - \mathbf{y}\|^2 = {} & \left\|\sum_{i=1}^N p_i\mathbf{x}_i - \sum_{i=1}^N q_i\mathbf{y}_i\right\|^2 = \left\|\sum_{i=1}^N p_i\mathbf{x}_i - \sum_{i=1}^N p_i\mathbf{y}_i + \sum_{i=1}^N p_i\mathbf{y}_i - \sum_{i=1}^N q_i\mathbf{y}_i\right\|^2 \\
\leq {} & 2\left\|\sum_{i=1}^N p_i(\mathbf{x}_i - \mathbf{y}_i)\right\|^2 + 2\left\|\sum_{i=1}^N (p_i - q_i)\mathbf{y}_i\right\|^2 \\
\leq {} & 2\left(\sum_{i=1}^N p_i\|\mathbf{x}_i - \mathbf{y}_i\|\right)^2 + 2\left(\sum_{i=1}^N |p_i - q_i|\|\mathbf{y}_i\|\right)^2 \\
\leq {} & 2\sum_{i=1}^N p_i\|\mathbf{x}_i - \mathbf{y}_i\|^2 + 2D^2\|\boldsymbol{p} - \boldsymbol{q}\|_1^2,
\end{aligned}$$

where the first inequality is due to $(a+b)^2 \leq 2a^2 + 2b^2$ for any $a, b \in \mathbb{R}$, and the last step is due to Cauchy-Schwarz inequality, Assumption 1 and the definition of ℓ_1-norm. ■

Proof [of Lemma 11] The proof follows a similar flow as Lemma 10. Specifically, we first decompose it as

$$\|\boldsymbol{w}-\boldsymbol{w}'\|_1^2 = \left\|\sum_{i=1}^N q_i \boldsymbol{p}_i - \sum_{i=1}^N q_i' \boldsymbol{p}_i'\right\|_1^2 \le 2\underbrace{\left\|\sum_{i=1}^N q_i(\boldsymbol{p}_i - \boldsymbol{p}_i')\right\|_1^2}_{\text{TERM (A)}} + 2\underbrace{\left\|\sum_{i=1}^N (q_i - q_i')\boldsymbol{p}_i'\right\|_1^2}_{\text{TERM (B)}}.$$

For TERM (A), we have

$$\begin{aligned}\text{TERM (A)} &= \left\|\sum_{i=1}^N q_i(\boldsymbol{p}_i - \boldsymbol{p}_i')\right\|_1^2 = \left(\sum_{i=1}^N \left|\sum_{j=1}^d q_i(p_{i,j} - p_{i,j}')\right|\right)^2 \le \left(\sum_{i=1}^N \sum_{j=1}^d q_i \left|p_{i,j} - p_{i,j}'\right|\right)^2 \\ &= \left(\sum_{i=1}^N q_i \sum_{j=1}^d \left|p_{i,j} - p_{i,j}'\right|\right)^2 \le \sum_{i=1}^N q_i \left(\sum_{j=1}^d \left|p_{i,j} - p_{i,j}'\right|\right)^2 = \sum_{i=1}^N q_i \|\boldsymbol{p}_i - \boldsymbol{p}_i'\|_1^2.\end{aligned}$$

For TERM (B), we have

$$\begin{aligned}\text{TERM (B)} &= \left\|\sum_{i=1}^N (q_i - q_i')\boldsymbol{p}_i'\right\|_1^2 = \left(\sum_{i=1}^N \left|\sum_{j=1}^d (q_i - q_i')p_{i,j}'\right|\right)^2 \le \left(\sum_{i=1}^N \sum_{j=1}^d |q_i - q_i'|\, p_{i,j}'\right)^2 \\ &= \left(\sum_{i=1}^N |q_i - q_i'| \sum_{j=1}^d p_{i,j}'\right)^2 = \left(\sum_{i=1}^N |q_i - q_i'|\right)^2 = \|\boldsymbol{q} - \boldsymbol{q}'\|_1^2,\end{aligned}$$

where the second last step is due to $\sum_{j=1}^d p_{i,j}' = 1$. Combining the bounds for TERM (A) and TERM (B) finishes the proof. ■

Proof [of Lemma 12] Denoting the optimal step size by $\eta_\star \triangleq \sqrt{\log(4XY)/X}$, if the optimal step size satisfies $\eta \le \eta_\star \le 2\eta$, where $\eta \le \eta_\star$ can be guaranteed if $C_0 \ge \frac{\sqrt{X}}{2T}$, then

$$\frac{1}{\eta}\log\frac{Y}{\eta^2} + \eta X \le \frac{2}{\eta_\star}\log\frac{4Y}{\eta_\star^2} + \eta_\star X \le 3\sqrt{X\log(4XY)}.$$

Otherwise, if the optimal step size is greater than the maximum step size in the parameter pool, i.e., $\eta_\star \ge (\eta = \eta_1 = \frac{1}{2C_0})$, then we have

$$\frac{1}{\eta}\log\frac{Y}{\eta^2} + \eta X \le \frac{1}{\eta}\log\frac{Y}{\eta^2} + \eta_\star X \le 2C_0\log(4YC_0^2) + \sqrt{X\log(4XY)}.$$

Overall, it holds that

$$\frac{1}{\eta}\log\frac{Y}{\eta^2} + \eta X \le 2C_0\log(4YC_0^2) + 4\sqrt{X\log(4XY)},$$

which completes the proof. ■

Proof [of Lemma 13] The proof follows the same flow as Lemma 12. ■

Proof [of Lemma 14] This result is inspired by Lemma 5 of Chen et al. (2023), and we generalize it to arbitrary variables for our purpose. Specifically, we consider two cases: $A < T$ and $A \geq T$. For the first case, if $A < T$, it holds that

$$\sum_{t=1}^{T} \frac{a_t}{bt} = \sum_{t=1}^{A} \frac{a_t}{bt} + \sum_{A+1}^{T} \frac{a_t}{bt} \leq \frac{a_{\max}}{b} \sum_{t=1}^{A} \frac{1}{t} + \frac{1}{b(A+1)} \sum_{A+1}^{T} a_t \leq \frac{a_{\max}}{b}(1 + \log A) + \frac{1}{b^2},$$

where the last step is due to $\sum_{A+1}^{T} a_t \leq \sum_{t=1}^{T} a_t \leq A/b$. The case of $A < T$ can be proved similarly, which finishes the proof. ■

Proof [of Lemma 15] The one-line proof is presented below:

$$\log(a+b) \leq \log(Ca+b) \leq \log(Ca) + \log\left(1 + \frac{b}{Ca}\right) \leq \log(Ca) + \frac{b}{C},$$

where the first step is due to $C \geq 1$, and the last step adopts $\log(1+x) \leq x$ for any $x \geq 0$. ■

E.2 Stability Analysis of Base Algorithms

In this part, we analyze the negative stability terms in the optimistic OMD analysis, for convex, exp-concave and strongly convex functions, respectively. For simplicity, we define the *empirical gradient variation* below:

$$\bar{V}_T \triangleq \sum_{t=2}^{T} \|\mathbf{g}_t - \mathbf{g}_{t-1}\|^2, \text{ where } \mathbf{g}_t \triangleq \nabla f_t(\mathbf{x}_t). \tag{E.1}$$

Next we provide the regret analysis in terms of the empirical gradient-variation $\bar{V}_T$, for strongly convex (Lemma 19), exp-concave (Lemma 20), and convex (Lemma 21) functions.

Lemma 19. *Under Assumptions 1, 2, and 3, if the loss functions are λ-strongly convex, OOGD (2.3) with $\mathbf{m}_t = \nabla f_{t-1}(\mathbf{x}_{t-1})$ and $\eta_t = 2/(\kappa + \lambda t)$, where κ is a parameter to be specified, enjoys the following empirical gradient-variation bound:*

$$\sum_{t=1}^{T} f_t(\mathbf{x}_t) - \min_{\mathbf{x}\in\mathcal{X}} \sum_{t=1}^{T} f_t(\mathbf{x}) \leq \frac{16G^2}{\lambda} \log\left(1 + \lambda \bar{V}_T\right) + \frac{1}{4}\kappa D^2 - \frac{\kappa}{8} \sum_{t=2}^{T} \|\mathbf{x}_t - \mathbf{x}_{t-1}\|^2 + \mathcal{O}(1).$$

Lemma 20. *Under Assumptions 1, 2, and 3, if the loss functions are α-exp-concave,* OOMD *(2.4) with $U_t = \kappa I + \frac{\alpha G^2}{2} I + \frac{\alpha}{2} \sum_{s=1}^{t-1} \nabla f_s(\mathbf{x}_s) \nabla f_s(\mathbf{x}_s)^\top$, where κ is a parameter to be specified, enjoys the following empirical gradient-variation bound:*

$$\sum_{t=1}^{T} f_t(\mathbf{x}_t) - \min_{\mathbf{x}\in\mathcal{X}} \sum_{t=1}^{T} f_t(\mathbf{x}) \leq \frac{16d}{\alpha} \log\left(1 + \frac{\alpha}{8\kappa d} \bar{V}_T\right) + \frac{1}{2}\kappa D^2 - \frac{\kappa}{4} \sum_{t=2}^{T} \|\mathbf{x}_t - \mathbf{x}_{t-1}\|^2 + \mathcal{O}(1).$$

Lemma 21. *Under Assumptions 1, 2, and 3, if the loss functions are convex, OOGD (2.3) with* $\mathbf{m}_t = \nabla f_{t-1}(\mathbf{x}_{t-1})$ *and* $\eta_t = \min\left\{D/\sqrt{1+\bar{V}_{t-1}}, 1/\kappa\right\}$*, where* κ *is a parameter to be specified, enjoys the following empirical gradient-variation bound:*

$$\sum_{t=1}^T f_t(\mathbf{x}_t) - \min_{\mathbf{x}\in\mathcal{X}} \sum_{t=1}^T f_t(\mathbf{x}) \le 5D\sqrt{1+\bar{V}_T} + \kappa D^2 - \frac{\kappa}{4}\sum_{t=2}^T \|\mathbf{x}_t - \mathbf{x}_{t-1}\|^2 + \mathcal{O}(1).$$

Proof [of Lemma 19] The proof mainly follows Theorem 3 of Chen et al. (2023). Following the almost the same regret decomposition in Lemma 21, it holds that

$$\sum_{t=1}^T f_t(\mathbf{x}_t) - \sum_{t=1}^T f_t(\mathbf{x}^\star) \le \underbrace{\sum_{t=1}^T \eta_t \|\nabla f_t(\mathbf{x}_t) - \mathbf{m}_t\|^2}_{\textsc{Adaptivity}} + \underbrace{\sum_{t=1}^T \frac{1}{\eta_t}\left(\mathcal{D}_\psi(\mathbf{x}^\star, \widehat{\mathbf{x}}_t) - \mathcal{D}_\psi(\mathbf{x}^\star, \widehat{\mathbf{x}}_{t+1})\right)}_{\textsc{Opt-Gap}}$$
$$\underbrace{- \sum_{t=1}^T \frac{1}{\eta_t}\left(\mathcal{D}_\psi(\widehat{\mathbf{x}}_{t+1}, \mathbf{x}_t) + \mathcal{D}_{\psi_t}(\mathbf{x}_t, \widehat{\mathbf{x}}_t)\right)}_{\textsc{Stability}} \underbrace{- \frac{\lambda}{2}\sum_{t=1}^T \|\mathbf{x}_t - \mathbf{x}^\star\|^2}_{\textsc{Negativity}}.$$

First, we analyze the optimality gap,

$$\begin{aligned}
\textsc{Opt-Gap} &\le \frac{1}{\eta_1}\mathcal{D}_\psi(\mathbf{x}^\star, \widehat{\mathbf{x}}_1) + \sum_{t=1}^T \left(\frac{1}{\eta_{t+1}} - \frac{1}{\eta_t}\right)\mathcal{D}_\psi(\mathbf{x}^\star, \widehat{\mathbf{x}}_{t+1}) \\
&\le \frac{1}{4}(\kappa+\lambda)D^2 + \frac{\lambda}{4}\sum_{t=1}^T \|\mathbf{x}^\star - \widehat{\mathbf{x}}_{t+1}\|^2.
\end{aligned}$$

We handle the last term by leveraging the negative term imported by strong convexity:

$$\begin{aligned}
\textsc{Opt-Gap} - \textsc{Negativity} &\le \frac{1}{4}(\kappa+\lambda)D^2 + \frac{\lambda}{4}\sum_{t=1}^T \|\mathbf{x}^\star - \widehat{\mathbf{x}}_{t+1}\|^2 - \frac{\lambda}{2}\sum_{t=1}^T \|\mathbf{x}_t - \mathbf{x}^\star\|^2 \\
&\le \frac{1}{4}\kappa D^2 + \frac{\lambda}{2}\sum_{t=1}^T \|\mathbf{x}_t - \widehat{\mathbf{x}}_{t+1}\|^2 + \mathcal{O}(1).
\end{aligned}$$

The second term above can be bounded by the stability of optimistic OMD:

$$\frac{\lambda}{2}\sum_{t=1}^T \|\mathbf{x}_t - \widehat{\mathbf{x}}_{t+1}\|^2 \le \frac{\lambda}{2}\sum_{t=1}^T \eta_t^2 \|\nabla f_t(\mathbf{x}_t) - \mathbf{m}_t\|^2 \le \sum_{t=1}^T \eta_t \|\nabla f_t(\mathbf{x}_t) - \mathbf{m}_t\|^2.$$

Finally, we lower-bound the stability term as

$$\begin{aligned}
\textsc{Stability} &= \sum_{t=1}^T \frac{\kappa+\lambda t}{4}\left(\|\widehat{\mathbf{x}}_{t+1} - \mathbf{x}_t\|^2 + \|\mathbf{x}_t - \widehat{\mathbf{x}}_t\|^2\right) && \left(\text{by } \eta_t = \tfrac{2}{\kappa+\lambda t}\right) \\
&\ge \frac{\kappa}{4}\sum_{t=2}^T \left(\|\widehat{\mathbf{x}}_t - \mathbf{x}_{t-1}\|^2 + \|\mathbf{x}_t - \widehat{\mathbf{x}}_t\|^2\right) \ge \frac{\kappa}{8}\sum_{t=2}^T \|\mathbf{x}_t - \mathbf{x}_{t-1}\|^2.
\end{aligned}$$

Choosing the optimism as $\mathbf{m}_t = \nabla f_{t-1}(\mathbf{x}_{t-1})$, we have

$$\textsc{Reg}_T \le 2\sum_{t=1}^T \eta_t \|\nabla f_t(\mathbf{x}_t) - \nabla f_{t-1}(\mathbf{x}_{t-1})\|^2 + \frac{1}{4}\kappa D^2 - \frac{\kappa}{8}\sum_{t=2}^T \|\mathbf{x}_t - \mathbf{x}_{t-1}\|^2 + \mathcal{O}(1).$$

To analyze the first term above, we follow the similar argument of Chen et al. (2023). By Lemma 14 with $a_t = \|\nabla f_t(\mathbf{x}_t) - \nabla f_{t-1}(\mathbf{x}_{t-1})\|^2$, $a_{\max} = 4G^2$, $A = \lceil \lambda \bar{V}_T \rceil$, and $b = \lambda$, it holds that

$$\sum_{t=1}^T \frac{1}{\lambda t}\|\mathbf{g}_t - \mathbf{g}_{t-1}\|^2 \le \frac{4G^2}{\lambda}\log\left(1 + \lambda \bar{V}_T\right) + \frac{4G^2}{\lambda} + \frac{1}{\lambda^2}.$$

Since $\eta_t = 2/(\kappa + \lambda t) \le 2/(\lambda t)$, combining existing results, we have

$$\textsc{Reg}_T \le \frac{16G^2}{\lambda}\log\left(1 + \lambda \bar{V}_T\right) + \frac{1}{4}\kappa D^2 - \frac{\kappa}{8}\sum_{t=2}^T \|\mathbf{x}_t - \mathbf{x}_{t-1}\|^2 + \mathcal{O}(1),$$

which completes the proof. ■

Proof [of Lemma 20] The proof mainly follows Theorem 15 of Chiang et al. (2012). Denoting by $\mathbf{x}^\star \in \arg\min_{\mathbf{x}\in\mathcal{X}} \sum_{t=1}^T f_t(\mathbf{x})$, it holds that

$$\sum_{t=1}^T f_t(\mathbf{x}_t) - \sum_{t=1}^T f_t(\mathbf{x}^\star) \le \underbrace{\sum_{t=1}^T \|\nabla f_t(\mathbf{x}_t) - \mathbf{m}_t\|^2_{U_t^{-1}}}_{\textsc{Adaptivity}} + \underbrace{\sum_{t=1}^T (\mathcal{D}_{\psi_t}(\mathbf{x}^\star, \widehat{\mathbf{x}}_t) - \mathcal{D}_{\psi_t}(\mathbf{x}^\star, \widehat{\mathbf{x}}_{t+1}))}_{\textsc{Opt-Gap}}$$
$$- \underbrace{\sum_{t=1}^T (\mathcal{D}_{\psi_t}(\widehat{\mathbf{x}}_{t+1}, \mathbf{x}_t) + \mathcal{D}_{\psi_t}(\mathbf{x}_t, \widehat{\mathbf{x}}_t))}_{\textsc{Stability}} - \underbrace{\frac{\alpha}{2}\sum_{t=1}^T \|\mathbf{x}_t - \mathbf{x}^\star\|^2_{\nabla f_t(\mathbf{x}_t)\nabla f_t(\mathbf{x}_t)^\top}}_{\textsc{Negativity}},$$

where the last term is imported by the definition of exp-concavity. First, the optimality gap satisfies

$$\begin{aligned}\textsc{Opt-Gap} &= \frac{1}{2}\sum_{t=1}^T \|\mathbf{x}^\star - \widehat{\mathbf{x}}_t\|^2_{U_t} - \frac{1}{2}\sum_{t=1}^T \|\mathbf{x}^\star - \widehat{\mathbf{x}}_{t+1}\|^2_{U_t} \\ &\le \frac{1}{2}\|\mathbf{x}^\star - \widehat{\mathbf{x}}_1\|^2_{V_1} + \frac{1}{2}\sum_{t=1}^T (\|\mathbf{x}^\star - \widehat{\mathbf{x}}_{t+1}\|^2_{U_{t+1}} - \|\mathbf{x}^\star - \widehat{\mathbf{x}}_{t+1}\|^2_{U_t}) \\ &\le \frac{1}{2}\kappa D^2 + \frac{\alpha G^2 D^2}{4} + \frac{\alpha}{4}\sum_{t=1}^T \|\mathbf{x}^\star - \widehat{\mathbf{x}}_{t+1}\|^2_{\nabla f_t(\mathbf{x}_t)\nabla f_t(\mathbf{x}_t)^\top}.\end{aligned}$$

We handle the last term by leveraging the negative term imported by exp-concavity:

$$\begin{aligned}&\textsc{Opt-Gap} - \textsc{Negativity} \\ &\le \frac{1}{2}\kappa D^2 + \frac{\alpha}{4}\sum_{t=1}^T \|\mathbf{x}^\star - \widehat{\mathbf{x}}_{t+1}\|^2_{\nabla f_t(\mathbf{x}_t)\nabla f_t(\mathbf{x}_t)^\top} - \frac{\alpha}{2}\sum_{t=1}^T \|\mathbf{x}_t - \mathbf{x}^\star\|^2_{\nabla f_t(\mathbf{x}_t)\nabla f_t(\mathbf{x}_t)^\top} + \mathcal{O}(1) \\ &\le \frac{1}{2}\kappa D^2 + \frac{\alpha}{2}\sum_{t=1}^T \|\mathbf{x}_t - \widehat{\mathbf{x}}_{t+1}\|^2_{\nabla f_t(\mathbf{x}_t)\nabla f_t(\mathbf{x}_t)^\top} + \mathcal{O}(1),\end{aligned}$$

where the local norm of the second term above can be transformed into U_t:

$$\frac{\alpha}{2}\sum_{t=1}^T \|\mathbf{x}_t - \widehat{\mathbf{x}}_{t+1}\|^2_{\nabla f_t(\mathbf{x}_t)\nabla f_t(\mathbf{x}_t)^\top} \le \frac{\alpha G^2}{2}\sum_{t=1}^T \|\mathbf{x}_t - \widehat{\mathbf{x}}_{t+1}\|^2 \le \sum_{t=1}^T \|\mathbf{x}_t - \widehat{\mathbf{x}}_{t+1}\|^2_{U_t}.$$

Using the stability of optimistic OMD (Chiang et al., 2012, Proposition 7), the above term can be further bounded by

$$\sum_{t=1}^T \|\mathbf{x}_t - \widehat{\mathbf{x}}_{t+1}\|^2_{U_t} \le \sum_{t=1}^T \|\nabla f_t(\mathbf{x}_t) - \mathbf{m}_t\|^2_{U_t^{-1}}.$$

By choosing the optimism as $\mathbf{m}_t = \nabla f_{t-1}(\mathbf{x}_{t-1})$, the above term can be consequently bounded due to Lemma 19 of Chiang et al. (2012):

$$\sum_{t=1}^T \|\nabla f_t(\mathbf{x}_t) - \nabla f_{t-1}(\mathbf{x}_{t-1})\|^2_{U_t^{-1}} \le \frac{8d}{\alpha}\log\left(1 + \frac{\alpha}{8\kappa d}\bar{V}_T\right).$$

The last step is to analyze the negative stability term:

$$\begin{aligned}\textsc{Stability} &= \sum_{t=1}^T (\mathcal{D}_{\psi_t}(\widehat{\mathbf{x}}_{t+1}, \mathbf{x}_t) + \mathcal{D}_{\psi_t}(\mathbf{x}_t, \widehat{\mathbf{x}}_t)) = \frac{1}{2}\sum_{t=1}^T \|\widehat{\mathbf{x}}_{t+1} - \mathbf{x}_t\|^2_{U_t} + \frac{1}{2}\sum_{t=1}^T \|\mathbf{x}_t - \widehat{\mathbf{x}}_t\|^2_{U_t} \\ &\ge \frac{\kappa}{2}\sum_{t=1}^T \|\widehat{\mathbf{x}}_{t+1} - \mathbf{x}_t\|^2 + \frac{\kappa}{2}\sum_{t=1}^T \|\mathbf{x}_t - \widehat{\mathbf{x}}_t\|^2 \ge \frac{\kappa}{4}\sum_{t=2}^T \|\mathbf{x}_t - \mathbf{x}_{t-1}\|^2.\end{aligned}$$

Combining existing results, we have

$$\textsc{Reg}_T \le \frac{16d}{\alpha}\log\left(1 + \frac{\alpha}{8\kappa d}\bar{V}_T\right) + \frac{1}{2}\kappa D^2 - \frac{\kappa}{4}\sum_{t=2}^T \|\mathbf{x}_t - \mathbf{x}_{t-1}\|^2 + \mathcal{O}(1),$$

which completes the proof. ■

Proof [of Lemma 21] The proof mainly follows Theorem 11 of Chiang et al. (2012). Following the standard analysis of optimistic OMD, e.g., Theorem 1 of Zhao et al. (2024),

$$\begin{aligned}\sum_{t=1}^T f_t(\mathbf{x}_t) - \sum_{t=1}^T f_t(\mathbf{x}^\star) \le{}& \underbrace{\sum_{t=1}^T \eta_t \|\nabla f_t(\mathbf{x}_t) - \mathbf{m}_t\|^2}_{\textsc{Adaptivity}} + \underbrace{\sum_{t=1}^T \frac{1}{\eta_t}(\mathcal{D}_\psi(\mathbf{x}^\star, \widehat{\mathbf{x}}_t) - \mathcal{D}_\psi(\mathbf{x}^\star, \widehat{\mathbf{x}}_{t+1}))}_{\textsc{Opt-Gap}} \\ & - \underbrace{\sum_{t=1}^T \frac{1}{\eta_t}(\mathcal{D}_\psi(\widehat{\mathbf{x}}_{t+1}, \mathbf{x}_t) + \mathcal{D}_{\psi_t}(\mathbf{x}_t, \widehat{\mathbf{x}}_t))}_{\textsc{Stability}},\end{aligned}$$

where $\mathbf{x}^\star \in \arg\min_{\mathbf{x}\in\mathcal{X}} \sum_{t=1}^T f_t(\mathbf{x})$ and $\psi(\cdot) \triangleq \frac{1}{2}\|\cdot\|^2$. The adaptivity term satisfies

$$\begin{aligned}\textsc{Adaptivity} &= \sum_{t=1}^T \eta_t \|\nabla f_t(\mathbf{x}_t) - \mathbf{m}_t\|^2 \le D\sum_{t=1}^T \frac{\|\nabla f_t(\mathbf{x}_t) - \nabla f_{t-1}(\mathbf{x}_{t-1})\|^2}{\sqrt{1 + \sum_{s=1}^{t-1}\|\nabla f_s(\mathbf{x}_s) - \nabla f_{s-1}(\mathbf{x}_{s-1})\|^2}} \\ &\le 4D\sqrt{1 + \sum_{t=1}^T \|\nabla f_t(\mathbf{x}_t) - \nabla f_{t-1}(\mathbf{x}_{t-1})\|^2} + 4DG^2,\end{aligned}$$

where the last step uses $\sum_{t=1}^{T} a_t / \sqrt{1 + \sum_{s=1}^{t-1} a_s} \leq 4\sqrt{1 + \sum_{t=1}^{T} a_t} + \max_{t\in[T]} a_t$ (Pogodin and Lattimore, 2019, Lemma 4.8). Next, we move on to the optimality gap,

$$
\begin{aligned}
\textsc{Opt-Gap} &= \sum_{t=1}^{T} \frac{1}{\eta_t}(\mathcal{D}_\psi(\mathbf{x}^\star, \widehat{\mathbf{x}}_t) - \mathcal{D}_\psi(\mathbf{x}^\star, \widehat{\mathbf{x}}_{t+1})) = \sum_{t=1}^{T} \frac{1}{2\eta_t}(\|\mathbf{x}^\star - \widehat{\mathbf{x}}_t\|^2 - \|\mathbf{x}^\star - \widehat{\mathbf{x}}_{t+1}\|^2) \\
&\leq \frac{\|\mathbf{x}^\star - \widehat{\mathbf{x}}_1\|^2}{2\eta_1} + \sum_{t=2}^{T} \left(\frac{1}{2\eta_t} - \frac{1}{2\eta_{t-1}}\right) \|\mathbf{x}^\star - \widehat{\mathbf{x}}_t\|^2 \\
&\leq \frac{D}{2}(1 + \kappa D) + D^2 \sum_{t=2}^{T} \left(\frac{1}{2\eta_t} - \frac{1}{2\eta_{t-1}}\right) \leq \frac{D}{2}(1 + \kappa D) + \frac{D^2}{2\eta_T} \\
&= \kappa D^2 + \frac{D}{2}\sqrt{1 + \bar{V}_T} + \mathcal{O}(1).
\end{aligned}
$$

Finally, we analyze the stability term,

$$
\begin{aligned}
\textsc{Stability} &= \sum_{t=1}^{T} \frac{1}{2\eta_t}(\|\widehat{\mathbf{x}}_{t+1} - \mathbf{x}_t\|^2 + \|\mathbf{x}_t - \widehat{\mathbf{x}}_t\|^2) \geq \sum_{t=2}^{T} \frac{1}{2\eta_t}(\|\widehat{\mathbf{x}}_t - \mathbf{x}_{t-1}\|^2 + \|\mathbf{x}_t - \widehat{\mathbf{x}}_t\|^2) \\
&\geq \sum_{t=2}^{T} \frac{1}{4\eta_t}\|\mathbf{x}_t - \mathbf{x}_{t-1}\|^2 \geq \frac{\kappa}{4}\sum_{t=2}^{T} \|\mathbf{x}_t - \mathbf{x}_{t-1}\|^2.
\end{aligned}
$$

Combining the above inequalities completes the proof. ∎